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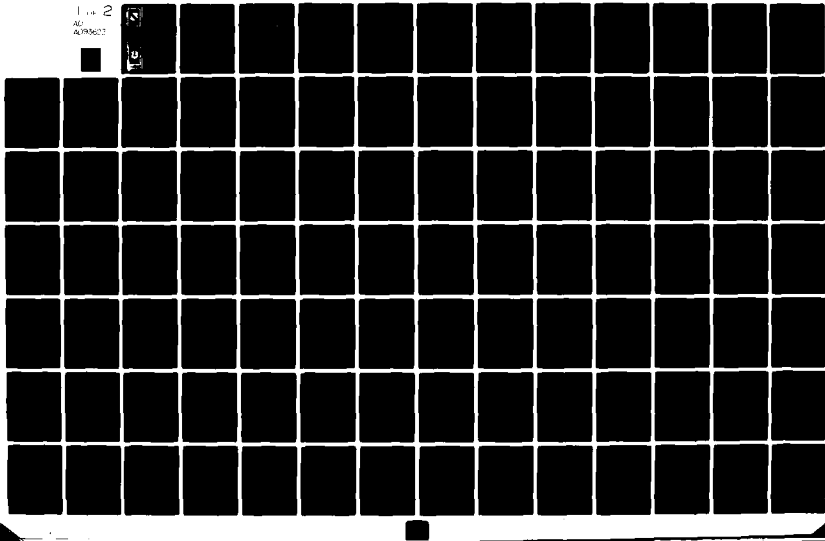

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# OPTIMAL PERIODIC CONTROL THEORY

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FINAL REPORT


RICHARD T. EVANS
LT COL, USAF




PROJECT 2304

AIR FORCE SYSTEMS COMMAND

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This report has been reviewed by the Commander and is releasable to the National Technical Information Service (NTIS). At NTIS it will be available to the general public, including foreign nations.

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Existing conditions for locally optimizing a periodic process with free period are corrected, extended, and clarified using the strong properties of the Hamiltonian structure of the Optimal Periodic Control (OPC) Problem and the relationships of the second variation that result from the problem's periodicity constraints. A neighboring optimum feedback controller is determined which regulates all perturbations back to the locally minimizing periodic trajectory. This periodic regulator extends many of the concepts



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of the time invariant regulator.

An illustrative problem is constructed to demonstrate the basic characteristics of this class of problem. A relatively comprehesive numerical investigation is conducted identifying a multiplicity of extremal solutions which are shown to form one-parameter families of solutions to the OPC. Bifurcation points, which define the critical periodic solutions common to intersecting families, are computed. Extremal solutions, satisfying all first order conditions, are tested for local sufficiency conditions by verifying the existance of the Ricatti' variable over one full period. The neighboring optimal feedback control law for a locally minimizing solution is tested by demonstrating the limit cycle behavior that results from perturbations to the initial conditions in a closed loop application.

An asymptotic series expansion is derived for the illustrative problem using a perturbation technique. An extremal solution in the form of a Fourier series, is obtained that accurately predicts the optimal period, the locally minimizing periodic solution, and the associated values of the Hamiltonian and the cost of the principal family.

# FORWARD

This report is, in essence, a dissertation submitted by Lt Col Richard T. Evans to the Faculty of the Graduate School of The University of Texas at Austin in partial fulfillment of the requirements for the degree of Doctor of Philosophy. This research was performed by the author under the supervision of Dr. Jason L. Speyer, Associate Professor in the Department of Aerospace Engineering and Engineering Mechanics, during the period June 1978 to August 1980.

This report also serves as the starting point for further research to be conducted by the author at the Frank J. Seiler Research Laboratory, United States Air Force Academy, Colorado under Work Unit 2304-F2-67, titled, "Optimal Periodic Control Applications in Aerospace Research."

I would like to express my sincere appreciation to all those who assisted me in this research effort and contributed to its successful completion.

I am particularly grateful to Dr. Jason L. Speyer, my dissertation supervisor, who initially inspired my interest in this research project and so generously provided his valuable advise, personal encouragement, and enthusiastic support during the entire effort. I am especially thankful to Dr. Roger A. Broucke, a member of my dissertation committee for his assistance and advise regarding the numerical determination of periodic solutions to fourth order dynamic systems. Also, I appreciate the recommendations and guidance offered by the remaining

members of my dissertation committee, Drs. Wallace T. Fowler, David G. Hull and Steven I. Marcus.

I express a special debt of gratitude to my long-time friend and colleague, Dr. Walter A. Figel, Jr., for reading the initial drafts of this dissertation and providing numerous critiques and recommendations. I would like to thank Linda Parrish who so expertly typed all drafts and the final copy of this manuscript. Also I am grateful to Charles Coker who skillfully produced several of the graphs in this work.

I appreciate the assistance and encouragement of numerous fellow graduate students and faculty of the Aerospace Engineering Department at the University of Texas. I am also quite pleased to acknowledge the Air Force Institute of Technology, Civilian Institute, which provided the opportunity for me to achieve this personal goal by sponsoring my entire graduate program.

Finally, I express my deepest gratitude to my wife, Mary, who daily provided the moral support, confidence, and encouragement that I needed to endure this program, and who frequently served as both mother and father to our daughter, Christine, and son, Richard.

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# CHAPTER 1

# INTRODUCTION

Optimal periodic control theory is an interesting and relatively unexplored branch of optimization theory. There are many applications identified in the literature and contributions to the theory are increasing. In the first section of this chapter preliminary remarks provide the general background and motivation for this research. After a statement of the problem and its assumptions, previous theoretical development is briefly reviewed. Then the objectives of this research and its scope are outlined. A description of the notation conventions employed concludes the chapter.

## 1.1 Preliminary Remarks

An optimal control problem, in general, can be reduced to a non-linear two-point boundary value problem in terms of the state variables and Lagrange multipliers. Except for relatively few cases, an analytical solution to the boundary value problem is untractable. One of the exceptions is the class of problems satisfied by the static equilibrium solution. For many practical applications, reasonable models of plant processes are approximately constants. Consequently, many control systems are operated "optimally" by regulating them as near as possible to an equilibrium state.

Improved performance frequently may be obtained by allowing the state of one of these regulated processes to vary with time. Notably in the field of Chemical Engineering, there are many applications studied in the literature [1, 2, 3]* in which periodic or cycling operation provides improved performance over steady state operation. This was the motivation for much of the early theoretical work in optimal periodic processes by Horn and Lin [4].

Another particularly interesting example is the result obtained by Speyer [5] in a controversy [5, 6, 7, 8] over an aerospace problem. He showed that the static cruise point is non-minimizing with respect to range factor (fuel rate per range rate) for several dynamic models of aircraft. He also showed numerically that operating along a cyclic but non-minimizing path provided better performance than operating at the static cruise point. He was not able to show what the optimum path was, however, and to date, this problem remains unsolved.

Later numerical attempts [9] to obtain a periodic solution to this problem using standard optimization techniques such as steepest ascent and conjugate gradient methods failed to converge. Comparable results obtained in the initial research for this work underscored the complexity of the particular application and motivated refocussing the present research effort on a more fundamental

*Numbers appearing in brackets refer to references listed in the bibliography.

and elucidating example. A specific problem has been constructed which is composed of very simple dynamics and a performance index allowing optimal periodic motion.

## 1.2 Statement of the Problem

The optimal periodic control problem is a subclass of optimum control problems that has two distinguishing features. First, the measure of performance (performance index or cost) is an average over one period of the performance criterion. Second, the periodicity constraint requires the initial and final states to be coincident. The following formulation and assumptions define the general optimal periodic control problem studied in this work.

Find the period, T, and the control, u, which minimize the performance index,

$$J = \frac{1}{T}\int_0^T L(x,u)dt, \tag{1.1}$$

subject to the constraints of the system, defined by

$$\dot{x} = f(x,u), \tag{1.2}$$

and to the periodicity condition applied at the boundary,

$$x(T) = x(0). \tag{1.3}$$

The assumptions applicable to this problem are:

(1) The period, $T \in (0,+\infty)$;

(2) The state vector, $x(t) \in X \subset R^n$, where X is the set of all permissible, continuous, T-periodic state functions in the n dimensional state space;

(3) The control vector, $u(t) \in U \subset R^m$, where U is the set of all permissible, piecewise continous, T-periodic control functions in the m dimensional input function space; and

(4) The functionals $f(\cdot,\cdot)$ and $L(\cdot,\cdot)$ and their first two derivatives with respect to both arguments are continuous. Note that the system, f, and the performance criterion, L, are both time invariant.

## 1.3 Historical Review

One of the earliest contributions to the theoretical development of optimal periodic control is the work of Horn and Lin [4]. They examine chemical engineering steady-state and batch processes that can be improved by cycling. Using a formulation for the problem similar to the previous section, they derive the first order necessary conditions for optimality. For the optimal period, they obtain the special form of the transversality condition which states that the performance index must equal the variational Hamiltonian evaluated along the extremal path. This special condition for optimal periodic processes has not been previously exploited in numerical procedures. Another early contributor to this theory is Fjeld [10].

The practical problem of establishing whether the optimal

steady state process can be improved by a suitable periodic process has been analyzed by Bittanti, Fronza, and Guardabassi [11]. They establish a new local sufficiency condition which involves transforming a second variation quadratic expression into the frequency domain and examining the result for non-negativity over the range of all positive frequencies. This frequency test along with other equivalent tests are presented in the work of Willems [12]. He intensively treats the optimal control of linear systems with respect to quadratic but not necessarily convex performance criteria.

The dynamic programming sufficiency conditions for the optimality of periodic control processes are developed in a later work by Maffezzoni [13]. Global conditions are derived for the optimal control from the Hamilton-Jacobi equation. Local sufficiency conditons for a weak minimum are derived by Bittanti, Locatelli, and Maffezzoni [14] for periodic processes with a given period. They use calculus of variations techniques similar to that of Bryson and Ho [15]. They also develop sufficient conditions for the existence of periodic solutions to the Riccati-type equations that result from the second variation analysis.

In more recent work, first order necessary conditions are reviewed by Gilbert [16, 17] for a very general formulation of the optimal periodic control problem. He develops relationships between elements of the solution sets for the optimal periodic control problem and optimal steady-state problem. Bernstein and Gilbert [18, 19] identify an overlooked normality condition in the development of

the frequency test [11]. They also expand the test to include the more general formulation of the problem above.

Much of the literature on optimal periodic control is reviewed by Guardabassi, Locatelli, and Rinaldi [20] in an important survey paper. Significant open aspects of the problem are identified, some of which are specifically addressed in this work. Another review by Bailey [21] emphasizes applications of the theory to a particular chemical engineering problem.

Important theoretical work from other disciplines should also be mentioned. There is considerable research devoted to the study of periodic solutions (orbits) for dynamic systems. They date back to at least 1890 and the extensive work by Poincarè [22] in celestial mechanics. With regard to finding periodic orbits, the most studied dynamic system is the restricted problem of three bodies. An excellent and thorough review of important work on this problem is provided by Szebehely [23].

Since his earlier work [24] on periodic orbits near equilibrium points of the restricted problem of three bodies, Broucke has meticulously studied periodic orbits of a large variety of fourth order dynamic systems. Much of this work is in preparation for publication. Two extensive studies of periodic solutions to other fourth order systems are by Contopoulos [25, 26] and Hénon [27].

A common element in all of these studies is the complexity of the resulting periodic solutions. Families of solutions exist which intersect at common critical solutions (bifurcation

points), and the solutions densely pack large regions of state space. Quasi-periodic solutions are also studied. This whole area of dynamics is currently engrossed in intense research efforts.

The text by Meirovitch [28] provides numerous special properties of Hamiltonian systems with periodic coefficients. Properties of the monodromy matrix, the system transition matrix evaluated over one period, are of particular interest to the present work.

### 1.4 Purpose and Scope

The main objective of this research is to develop a useful understanding of the structure and characteristics of periodic processes resulting from the optimal periodic control problem. An attempt has been made in this work to integrate applicable portions of the theory and related experience developed in the fields of analytical dynamics and celestial mechanics with that existing for this problem derived under the discipline of optimal control theory. It is hoped that this expansion of the theory will lead to its increased application and, in particular, to the development of improved numerical techniques for analyzing more complex systems.

There are three basic parts that comprise this study. The first part involves the general theoretical background of the problem. Necessary conditions, sufficient conditions, and tests for the optimality of continuous, time-invariant, periodic, control

systems are presented. Special properties, studied by analytical dynamicists, of time-invariant, Hamiltonian systems with periodic constraints are reviewed. Of particular importance are the properties of the monodromy matrix, a constant matrix determined by the transition matrix for a periodic Hamiltonian system evaluated over one period. Concluding the first part, contributions to the general theory developed during the present research are derived. Included is a new algebraic Riccati equation. It determines the initial values for periodic solutions to the Riccati differential equation generated from a second variation analysis. A similarity transformation of the monodromy matrix separates the system eigenvalues into the two diagonal blocks of the resulting matrix. This results in the necessary condition for periodic extremal solutions that corresponding monodromy matrices have no distinct eigenvalues on the unit circle. Local sufficiency conditions are then extended to the free period case, and finally, properties of a periodic regulator are developed.

In the second part of the study, a particular control problem is defined that assures the existence of optimal periodic solutions for some range of values of a parameter weighting the control. A fairly comprehensive numerical investigation of this control problem is conducted. Periodic solutions to the Euler-Lagrange equations resulting from the first order necessary conditions are computed. They form one parameter families which intersect at common "critical" solutions or bifurcation points. Several solutions are examined

along and between families to identify characteristic differences that distinguish the families. The necessary condition for an optimal period is applied which is satisfied by many of the solutions. Sufficiency conditions are checked for these solutions by integrating the corresponding matrix Riccati equation over one period to determine whether the Riccati variable exists during the entire interval. Finally, a neighboring optimum feedback control system is developed to operate near the optimal periodic process. The initial state is perturbed, demonstrating the behavior of the periodic regulator.

The final part of the study is the development of an approximate analytical solution for the particular control problem previously defined. A perturbation technique is used that results in a solution in the form of an asymptotic series expansion. The necessary condition for optimal period is applied leading to expressions in the perturbation parameter and the time for the optimal path, period, control, and performance index. These results are compared to a linear analysis near the static solution and to earlier numerical results obtained at the minimum solution.

Conclusions and recommendations for further research complete the study.

## 1.5 Notation Conventions

Set theory notation is used sparingly in this study. Symbols used and their intended definitions are listed below:

(1) $\in$ signifies "is an element of",

(2) $\subset$ signifies "is a subset of", and

(3) $R^n$ signifies a real, Euclidean n-dimensional hyperspace.

Thus the notation $T \in (0,+\infty)$ means that T is an element of the open set $0 < T < +\infty$, and the notation $x(\cdot) \in X \subset R^n$ means that x is an n-vector of the set of n-vectors X which is a subset of all vectors defined in n-dimensional, real, Euclidean hyperspace.

Vector and matrix notation are used throughout this study. The following conventions are used:

(1) Vectors are generally indicated by lower case letters and, unless otherwise indicated, are column vectors. Components of vectors are denoted by subscripts. Thus, the n-vector x and its components are:

$$x \equiv \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}.$$

(2) Matrices are indicated by upper case letters and sometimes by elements enclosed in brackets. Elements of matrices are denoted by double subscripts. Thus, the n row by m column matrix A and its elements are denoted by

$$A \equiv \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}.$$

(3) Scalars are easily recognized as such, otherwise they are specifically identified when used. Examples are the independent variables, $t$ and $\tau$, for time and extended time, and the indexing variables $i$, $j$, and $k$. Others are the performance index, $J$, and the variational Hamiltonian, $H$.

(4) The transpose of a vector or matrix is denoted by the superscript T, thus $A^T$ denotes the transpose of matrix A.

(5) The inverse of a square matrix is denoted by a superscript -1, thus $B^{-1}$ denotes the inverse of matrix B.

(6) The first partial derivative of a scalar with respect to a vector is a row vector designated by

$$H_x \equiv \frac{\partial H}{\partial x} \equiv \left[ \frac{\partial H}{\partial x_1} \quad \frac{\partial H}{\partial x_2} \cdots \frac{\partial H}{\partial x_n} \right].$$

(7) The second partial derivative of a scalar with respect to a vector is a matrix designated by

$$H_{xu} \equiv \frac{\partial}{\partial u}\left(\frac{\partial H}{\partial x}\right)^T \equiv \begin{bmatrix} H_{x_1 u_1} & \cdots & H_{x_1 u_m} \\ \vdots & & \vdots \\ H_{x_n u_1} & \cdots & H_{x_n u_m} \end{bmatrix}.$$

Note that the transpose of the matrix $H_{xu}$ is $H_{ux}$.

(8) The first partial derivative of a vector with respect to another vector is a matrix designated by

$$f_x \equiv \frac{\partial f}{\partial x} \equiv \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}.$$

(9) The first partial derivative of a matrix with respect to a vector is designated by

$$f_{xu} \equiv \frac{\partial^2 f_i}{\partial u_k \partial x_j}, \text{ where } i \text{ and } j = 1, \ldots, n \text{ and } k = 1, \ldots, m.$$

Other notation conventions used in this study are listed below:

(1) The dot convention is used to designate a total derivative with respect to the independent variable, generally time. Thus the following notation is equivalent,

$$\dot{x} \equiv \frac{dx}{dt}.$$

(2) The derivative operator, $D^n$, is used in chapter six to designate the $n^{th}$ derivative with respect to the expanded time, $\tau$,

$$\frac{d^2x}{d\tau^2} \equiv D^2x.$$

(3) Various notation conventions are used in the literature to identify the variations of a function or functional. In this study only the variation operators d and $\delta$ are used, conforming to the

predominant usage present in the literature. Unfortunately d is also used to denote a differential along the path which provides a source of confusion in the literature. The distinction between the variation operators is that d represents a total variation while δ represents a variation with time held constant. The following expressions provide the functional relationship between them; figure 1.1 graphically identifies this relationship,

$$\begin{aligned} dx(t_f) &\equiv x(t_f+dt_f) - x^o(t_f), \\ &= x(t_f) + \dot{x}(t_f)dt_f + \tfrac{1}{2}\ddot{x}(t_f)dt_f^2 + \ldots - x^o(t_f), \\ &= \delta x(t_f) + \dot{x}^o(t_f)dt_f + \text{higher orders}. \end{aligned} \quad (1.4)$$

In equation (1.4) and figure 1.1, $x^o$ is the extreme (reference) trajectory, x is a neighboring (comparison) path, and $\dot{x}^o$ is the result, to first order, of expanding $\dot{x}$ in a Taylor series about $\dot{x}^o$. In later chapters the superscript "o" designates the optimal trajectory or optimal solution.

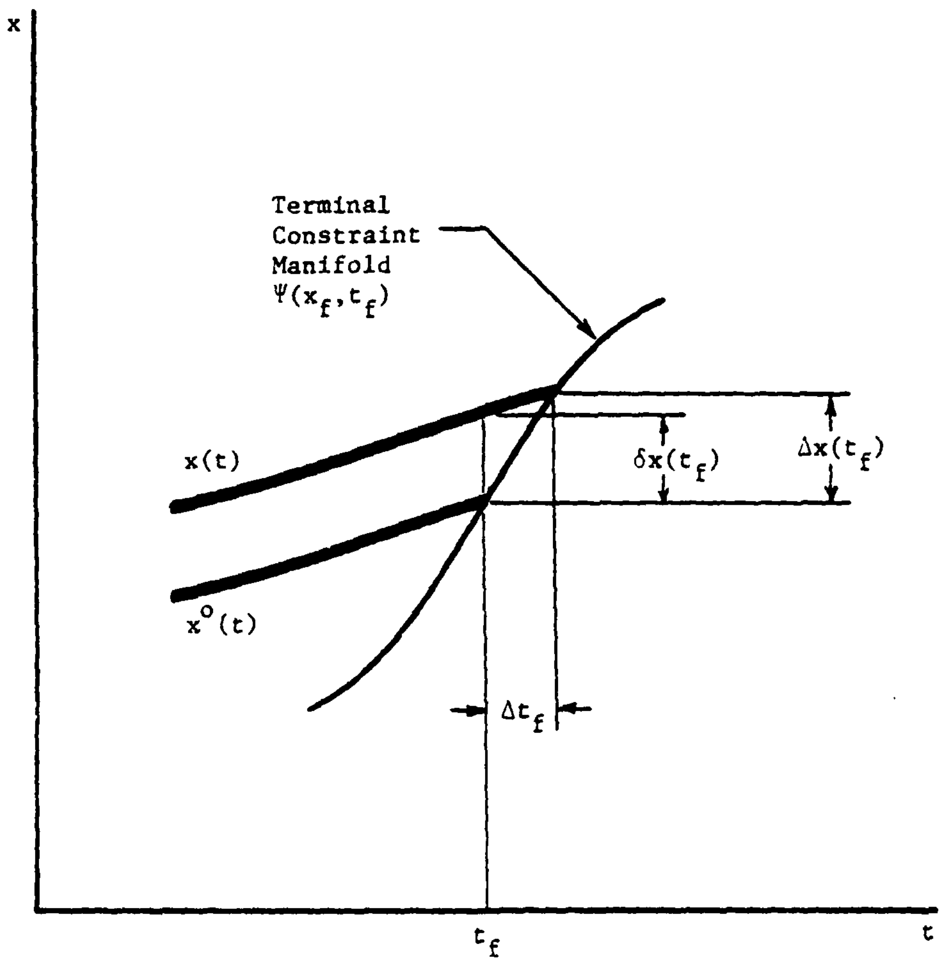


Figure 1.1 RELATIONSHIP BETWEEN VARIATIONS AT TERMINAL MANIFOLD

## CHAPTER 2

## CONDITIONS FOR OPTIMALITY

The conditions for optimality of a control process may be derived by at least two different methods. The classical calculus of variations approach is used in many texts [15, 29, 30, 31] to derive the conditions for local optima; whereas, Bellman's dynamic programming technique [32, 33, 34], using the Hamilton-Jacobi theory, also provides global results. In the first section of this chapter the existing first order necessary conditions for optimal periodic processes are derived using the calculus of variations. The next two sections review existing sufficient conditions for optimality derived from each method. The frequency test, which determines whether or not a periodic process can improve steady-state performance in a control system, is presented in the final section.

### 2.1 First Order Necessary Conditons

The optimal periodic control problem defined in the first chapter by equations (1.1) through (1.3) is rewritten here for convenience. Find the period, T, and the control, u, which minimize the performance index,

$$J = \frac{1}{T}\int_0^T L(x,u)dt, \tag{1.1}$$

subject to the following system constraint,

$$\dot{x} = f(x,u), \tag{1.2}$$

and the periodicity condition applied at the boundary,

$$x(T) = x(0). \tag{1.3}$$

The system constraint and periodic boundary conditions can be adjoined to the performance index with Lagrange multipliers forming the augmented performance index,

$$\bar{J} \equiv \nu^T \Psi(x(T),x(0)) + \frac{1}{T}\int_0^T [H(x,u,\lambda)-\lambda^T\dot{x}]dt. \tag{2.1}$$

The new functions introduced by equation (2.1) are the variational Hamiltonian,

$$H(x,u,\lambda) \equiv L(x,u) + \lambda^T f(x,u), \tag{2.2}$$

and the boundary condition null identity,

$$\Psi(x(T),x(0)) \equiv x(T) - x(0). \tag{2.3}$$

The n-vector Lagrange multipliers, $\nu$ and $\lambda$, multiply the null identities formed from the boundary condition equation (1.3) and the system constraint equation (1.2) respectively.

First order necessary conditions for the optimal periodic control are now derived. From fundamental arguments in the calculus of variations the first variation of the augmented performance index must vanish. Written as the sum of three terms, the first variation is

$$d\bar{J} = d(v^T\Psi) + d(\tfrac{1}{T})\int_0^T [H-\lambda^T\dot{x}]dt + \tfrac{1}{T}d(\int_0^T [H-\lambda^T\dot{x}]dt). \quad (2.4)$$

Expanding the initial term of the first variation provides

$$d(v^T\Psi) = dv^T\Psi + v^T[\Psi_{x(T)}dx(T) + \Psi_{x(0)}dx(0)].$$

This may be simplified by recognizing, from equation (2.3), that $\Psi_{x(T)} = 1$ and $\Psi_{x(0)} = -1$. Substituting these into the above expression for the first term gives

$$d(v^T\Psi) = dv^T\Psi + v^T[dx(T) - dx(0)]. \quad (2.5)$$

Operating as indicated on the second term of equation (2.4) yields

$$d(\tfrac{1}{T})\int_0^T [H-\lambda^T\dot{x}]dt = -\frac{dT}{T^2}\int_0^T [H-\lambda^T\dot{x}]dt .$$

Simplifying this by using equations (1.1), (1.2), and (2.2) gives

$$d(\tfrac{1}{T})\int_0^T [H-\lambda^T\dot{x}]dt = -\frac{dT}{T}J. \quad (2.6)$$

The expansion of the last term of equation (2.4) is common to free terminal time optimal control problems,

$$\tfrac{1}{T}d(\int_0^T [H-\lambda^T\dot{x}]dt) = \tfrac{1}{T}(H-\lambda^T\dot{x})dT$$

$$+ \frac{1}{T}\int_0^T [H_x\delta x + H_u\delta u + \delta\lambda^T(H_\lambda^T-\dot{x}) - \lambda^T\delta\dot{x}]dt.$$

Integrating the last term of the right hand integral by parts and using the relationship (valid to first order), $\delta x = dx - \dot{x}dt$, reduces the previous equation to

$$\frac{1}{T}d\left(\int_0^T [H-\lambda^T\dot{x}]dt\right) = \frac{1}{T}HdT + \frac{1}{T}(\lambda^T dx)\Bigg|_{t=0}^{t=T}$$

$$+ \frac{1}{T}\int_0^T [(H_x+\dot{\lambda}^T)\delta x + H_u\delta u + \delta\lambda^T(H_\lambda^T-\dot{x})]dt \ . \qquad (2.7)$$

Combining the results for the three terms given by equations (2.5), (2.6), and (2.7) provides the following expression for the first variation,

$$d\bar{J} = d\nu^T\Psi + \frac{1}{T}(H-J)dT + [(\nu^T + \frac{1}{T}\lambda^T(t))dx(t)]\Bigg|_{t=0}^{t=T}$$

$$+ \frac{1}{T}\int_0^T [(H_x+\dot{\lambda}^T)\delta x + H_u\delta u + \delta\lambda^T(H_\lambda^T-\dot{x})]dt. \qquad (2.8)$$

The criteria that the first variation must be zero when evaluated along an extremal path will now be examined. The terms which include the Lagrange multiplier variations vanish since their coefficients are the null identities previously defined. Recognizing that using Lagrange multipliers permits treating all other variables as though they were independent allows the remaining coefficients of variations in equation (2.8) to be isolated by appropriately selecting

admissable comparison paths. Thus each of these coefficients must also equate to zero when evaluated along an extremal path.

The Euler-Lagrange equations result from requiring the coefficients of the variations in the integral of equation (2.8) to vanish. They are:

$$\dot{x} = H_\lambda^T = f(x,u), \tag{2.9}$$

$$\dot{\lambda} = -H_x^T = -f_x^T(x,u)\lambda - L_x^T(x,u), \tag{2.10}$$

$$0 = H_u^T = f_u^T(x,u)\lambda + L_u^T(x,u). \tag{2.11}$$

The fixed point relationships are established by equating the remaining coefficients of the equation to zero. The coefficient in the first term is the boundary null identity and gives back the prescribed boundary conditions,

$$x(T) = x(0). \tag{2.12}$$

The natural boundary conditions, i.e., transversality condition, comes from the remaining terms,

$$\lambda(T) = \lambda(0) = -T\nu, \tag{2.13}$$

$$H(T) = H(0) = J^o. \tag{2.14}$$

Any periodic solution to the two point boundary value problem, equations (2.9) through (2.13), is an extremum of the problem

when the period is specified. The condition (2.14) relating the variational Hamiltonian and the performance index, evaluated along the optimal path is the special condition, first derived by Horn and Lin [4], for testing the optimal period. For the case where H is autonomous, which is assumed here, the Hamiltonian is a constant on any extremal path, and equation (2.14) is valid for all times. This condition is obviously not affected by the particular choice of the initial time.

## 2.2 Sufficient Conditions for a Local Minimum

A set of sufficient conditions for local optimality of the periodic control problem, defined by equations (1.1), (1.2), and (1.3) with the period T given, is reviewed. These conditions were initially derived by Bittanti, Locatelli, and Maffezzoni [14] using first and second variations and properties of periodic functions. In the following restatement of the sufficient conditions, the function $\Phi_A(T,0)$ is the transition matrix for the system $\dot{y} = Ay$ evaluated over one period, and the control $u^o(\cdot)$ is a piecewise-continuous, T-periodic function in the input function space. The superscript "o" designates the optimum solution.

The control $u^o(\cdot)$ is a local minimum if the following conditions are satisfied:

(1) The Euler-Lagrange equations, (2.9) through (2.11), the prescribed boundary condition, (2.12), and the transversality condition, (2.13), are satisfied; $u^o(\cdot)$ is a solution to (2.11) for $0 \leq t \leq T$, and

$x^o(\cdot)$ and $\lambda^o(\cdot)$ are the corresponding solutions to the resulting two-point boundary value problem;

(2) No eigenvalue of $\Phi_{f_x^o}(T,0)$ is equal to one;

(3) The strong form of the Legendre condition is satisfied,

$$H_{uu}^o > 0, \text{ for } 0 \le t \le T;$$

(4) A bounded symmetric solution to the following Riccati-type equation exists for $0 \le t \le T$,

$$\dot{P} = - PA - A^T P + PBP + C, \tag{2.15}$$

where $A = f_x^o - f_u^o(H_{uu}^o)^{-1}H_{ux}^o$,

$$B = f_u^o(H_{uu}^o)^{-1}f_u^{oT}, \text{ and}$$

$$C = -H_{xx}^o + H_{ux}^{o\,T}(H_{uu}^o)^{-1}H_{ux}^o,$$

subject to the periodic boundary condition

$$P(T) = P(0); \text{ and} \tag{2.16}$$

(5) No eigenvalue of $\Phi_Z(T,0)$ is equal to one, where

$$Z = f_x^o - f_u^o(H_{uu}^o)^{-1}(H_{ux}^o + f_u^{oT}P). \tag{2.17}$$

It will be shown in chapter four that the last condition (5) as stated in [14], is never true since $\Phi_Z(T,0)$ always has a unit eigenvalue. A necessary condition for optimality is derived from the

eigenvalues of this matrix. Also, an algebraic Riccati equation is determined that provides the initial conditions for periodic solutions to equation (2.15) in condition (4).

## 2.3 Hamilton-Jacobi Theory

By extending the Hamilton-Jacobi theory to periodic optimization problems, Maffezzoni [13] derived a sufficient condition for optimality of the problem defined by equations (1.1), (1.2), and (1.3) with the period, T, free. It is assumed that the solution V(t,x) to the Hamilton-Jacobi equation,

$$V_t(t,x) + \min_{u \in U} H(x, V_x(t,x), u) = 0, \tag{2.18}$$

is twice continuously diffentiable with respect to both of its arguments throughout phase space. It is also assumed that the variational Hamiltonian is regular, i.e., that it has a unique minimum with respect to the control for given values of its arguments. With these assumptions, a statement of the sufficient condition follows.

If a real function $C(\cdot)$ exists relating solutions to (2.18) in the form

$$V(0,x) - V(T,x) = C(T), \tag{2.19}$$

and $u^*(x, V_x(t,x))$ is the control that minimizes the Hamiltonian in equation (2.18), for all $x \in X$, then the optimal control relative to x and the specified T is

$$u^o(t) = u^*(x^o(t), V_x(x^o(t),t)), \tag{2.20}$$

where $u^o$ and $x^o$ denote the optimal control and state relative to the period.

A corollary relationship to the previous condition is also developed in this work. Under the previous assumptions the optimal performance relative to x and T can be expressed as

$$J^o = \frac{V(0,x^o) - V(T,x^o)}{T} = \frac{C(T)}{T}. \tag{2.21}$$

## 2.4 The Frequency Test

The question of whether the optimal steady-state operation of a given plant can be improved by a cycling or periodic process is an important preliminary concern. This question has been considered in much of the previous work on optimal periodic control processes.

From a relaxed steady-state control theory approach, violation of the maximum principle by the optimal steady state control leads to an optimal solution of the bang-bang or chattering type. Reference should be made to Horn and Lin [4], Bailey and Horn [1], and Gilbert [16, 35] for further discussion and examples.

A second variation analysis by Bittanti, Fronza, and Guardabassi [11] established a frequency domain local sufficiency condition for determining whether the steady state control of a system can be improved by cycling. However, the condition does not provide sufficient information to determine what cyclic or periodic process is best. An overlooked normality (controllability) condition was identified by Bernstein and Gilbert [18] in a somewhat more general treatment of the same problem.

The frequency test is stated here without proof. For the optimal periodic control problem defined by equations (1.1), (1.2), and (1.3), where the system is assumed to be controllable, let

$$A \equiv f_x(x^o,u^o), \qquad B \equiv f_u(x^o,u^o),$$

$$P \equiv H_{xx}(x^o,u^o,\lambda^o), \qquad Q \equiv H_{xu}(x^o,u^o,\lambda^o),$$

$$R \equiv H_{uu}(x^o,u^o,\lambda^o), \tag{2.22}$$

and no eigenvalue of A has a zero real part. Willems allows any matrix A in his work [12]. The superscript on x, u, and $\lambda$ designates the optimal solution in the class of constant solutions of the optimal periodic control problem. Form the (n x n)-Hermitian matrix,

$$\Pi(\omega) = G^T(-j\omega)PG(j\omega) + Q^TG(j\omega) + G^T(-j\omega)Q + R, \tag{2.23}$$

where $G(s) \equiv (sI-A)^{-1}B$, $\omega$ is frequency, and $j = \sqrt{-1}$. If $\Pi(\omega)$ is not negative definite for all $\omega$, then the optimal control for the problem is not constant, but belongs to the class of time varying periodic controls.

# CHAPTER 3

# HAMILTONIAN SYSTEMS WITH PERIODIC SOLUTIONS

The optimal periodic control problem can be expressed in the standard canonical form of a Hamiltonian system by functionally eliminating the control from the Euler-Langrange equations. The transition matrix of this Hamiltonian system evaluated over one or an integral multiple of the period and the corresponding eigenvalues of this matrix have properties which are exploited in the theoretical developments of the next chapters. These important properties, as well as the fundamental properties of the Hamiltonian systems from which they are derived, are reviewed in this chapter. There are a vast number of references available that address different portions of the material covered here in various degrees of completeness. The intent of this chapter is to include in one place, in a consistent notation, a statement and derivation of properties that are fundamental to new theory developed in later chapters of this work.

## 3.1 Hamiltonian Structure of the Problem

In general the optimal periodic control problem can be expressed in the familiar canonical form of Hamilton's equations of motion. The necessary condition for all extrema, that the first variation of the performance index vanish, yields the Euler-Lagrange equations,

$$\dot{x} = H_{\lambda}^{T}(x,u,\lambda), \tag{3.1}$$

$$\dot{\lambda} = -H_{x}^{T}(x,u,\lambda), \tag{3.2}$$

$$0 = H_{u}^{T}(x,u,\lambda), \tag{3.3}$$

subject to the periodicity conditions at the boundary, where T is the period,

$$x(T) = x(0) \text{ and } \lambda(T) = \lambda(0). \tag{3.4}$$

An expression for the control, u, in terms of the state variables and Lagrange multipliers, x and $\lambda$, may be determined (at least implicitly) from the last of the Euler-Lagrange equations (3.3). Eliminating the control from the remaining two equations reduces the system of equations to the desired canonical form,

$$\dot{x} = H_{\lambda}^{T}(x,u(x,\lambda),\lambda), \tag{3.5}$$

$$\dot{\lambda} = -H_{x}^{T}(x,u(x,\lambda),\lambda). \tag{3.6}$$

In terms of the dynamicist, the variational Hamiltonian $H(x,\lambda)$ and the two n-vectors of the state, x, and the Lagrange multipliers, $\lambda$, are respectively the Hamiltonian function and the two phase space n-vectors of coordinates and momenta.

It is convenient to make a further simplification. First define the (2n x 2n) skew symmetric matrix, K, which is commonly called the fundamental symplectic matrix,

$$K \equiv \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \tag{3.7}$$

where I is the n square identity matrix. By inspection, the fundamental symplectic matrix satisfies

$$K^{-1} = K^T = -K. \tag{3.8}$$

Now Hamilton's equations may be written in the simplified form

$$\dot{y} = KH_y^T = Y(y), \tag{3.9}$$

where y is the 2n vector composed of the phase space components x and $\lambda$ and is subject to the periodicity condition

$$y(T) = y(0). \tag{3.10}$$

From equation (3.9) it is apparent that the tangent to a phase space trajectory is a function of the position vector, y, only. Hence, there is only one trajectory through a point in phase space, and each trajectory is fixed. It will be convenient later to identify these trajectories (periodic solutions to the Euler-Lagrange equations) by their initial conditions. Note that the system is assumed autonomous; that is, the Hamiltonian is not an explicit function of the time.

Another well known property of an autonomous system is that the Hamiltonian is a constant of the motion. This is easily shown by substituting in the expression for the time derivative of the Hamiltonian,

$$\dot{H}(x,u,\lambda) = H_x\dot{x} + H_\lambda\dot{\lambda} + H_u\dot{u}, \tag{3.11}$$

the equations of motion, (3.1) through (3.3), which gives

$$\dot{H}(x,\lambda) = H_x H_\lambda^T + H_\lambda(-H_x^T) . \tag{3.12}$$

Since the scalar inner product is invariant to the transpose, equation (3.12) reduces to

$$\dot{H} = 0, \tag{3.13}$$

and the Hamiltonian is a constant of the motion.

The first integral may be used to reduce by one the order of the differential equations that define the system. Any variables in the Hamiltonian function is expressable in terms of a constant parameter and the remaining variables. Given any set of initial conditions the constant parameter is uniquely determined. A further reduction in the order of the system is possible by eliminating the independent variable, time, from the equations. The computational scheme employed in this work to determine periodic solutions to the Euler-Lagrange equations exploits the first of these relationships and in effect reduces the degree of the problem by one order.

## 3.2 Symplectic Properties and the Transition Matrix

Some fundamental characteristics of all Hamiltonian systems are associated with the properties of symplectic matrices. To begin this section, the definitions of two types of symplectic matrices and

their corresponding properties are provided. Then the transition matrix of a Hamiltonian system, along with some of its basic properties, are derived. Finally, it is shown that the transition matrix of a Hamiltonian system is symplectic.

First, the (2n x 2n) matrix, B, is defined to be a symplectic matrix if

$$BKB^T = K, \tag{3.14}$$

where K is the fundamental symplectic matrix (3.7) defined in the previous section. Second, the (2n x 2n) matrix, C, is a skew symplectic matrix if

$$KCK^{-1} = -C^T. \tag{3.15}$$

With the use of the symplectic matrix relationships expressed in (3.8) many equivalent forms of (3.14) and (3.15) may be generated.

To illustrate properties associated with these matrices, partition each matrix, B and C, into four (n x n) submatrices which then can be expressed as

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}. \tag{3.16}$$

Substituting the partitioned matrices (3.16) into equations (3.14) and (3.15) provides the following properties for the symplectic matrix B,

$$B_{11}B_{12}^T = (B_{11}B_{12}^T)^T, \tag{3.17}$$

$$B_{22}B_{21}^T = (B_{22}B_{21}^T)^T, \tag{3.18}$$

$$B_{11}B_{22}^T - B_{12}B_{21}^T = I, \tag{3.19}$$

and for the skew symplectic matrix C,

$$C_{12} = C_{12}^T, \tag{3.20}$$

$$C_{21} = C_{21}^T, \tag{3.21}$$

$$C_{11}^T = -C_{22}. \tag{3.22}$$

One important result of equation (3.22) is that the trace of a skew symplectic matrix is zero.

Returning to the Hamiltonian system, first write its linear variational equations. This may be done by considering the result of a small perturbation, $\delta y$, from a periodic solution, y, to the equations of motion (3.9). Substituting the perturbed solution into the equations of motion gives

$$\dot{y} + \delta\dot{y} = KH_y^T(y+\delta y), \tag{3.23}$$

which may be expanded in a Taylor series about the nominal solution, y, resulting in

$$\dot{y} + \delta\dot{y} = K[H_y^T + H_{yy}\delta y + \text{order}(\delta y^2)]. \tag{3.24}$$

Neglecting $\delta y$ terms of order two and higher and using equation (3.9)

in the result gives the familiar form of the linear variational equations,

$$\delta\dot{y} = A(t)\delta y, \tag{3.25}$$

where $A(t) = KH_{yy}$ and the matrix of second partials, $H_{yy}$, is evaluated on the nominal solution y(t). Note that the matrix $H_{yy}$ is a real, symmetric matrix with each element either periodic of period T or constant. Note also that the matrix A(t) is skew symplectic as may be shown by substituting the matrix into equation (3.15) and reducing the result to the identity below:

$$\begin{aligned} K(KH_{yy})K^{-1} &= -(KH_{yy})^T \\ -H_{yy}K^{-1} &= -H_{yy}K^T \\ -H_{yy}K^T &= -H_{yy}K^T. \end{aligned} \tag{3.26}$$

The relationships for K expressed in equation (3.8) and the symmetry of $H_{yy}$ were used in the above equations (3.26).

A fundamental solution matrix, $\phi$, may be constructed by forming a square matrix composed of 2n columns of independent solutions to the variational equation (3.25). The solutions are real and in general not periodic. This matrix is also not unique; the relationship between any two fundamental matrices being

$$\phi_1(t) = \phi_2(t)C, \tag{3.27}$$

where C is a constant matrix for all t. The fundamental matrix, $\phi_1$, is called a principal fundamental matrix when the constant matrix

is the inverse of $\phi_2$ evaluated at some reference time $t_0$. This very useful, special case of $\phi_1$ is also called the transition matrix, typically expressed as $\Phi(t,t_0)$. Rewriting equation (3.27) in the following form defines the transition matrix,

$$\Phi(t,t_0) \equiv \phi(t)\phi^{-1}(t_0). \tag{3.28}$$

Note at the initial time, $t = t_0$, the transition matrix reduces to the identity matrix.

From its construction it is obvious that the transition matrix satisfies the variational equation (3.25), thus

$$\dot{\Phi}(t,t_0) = A(t)\Phi(t,t_0). \tag{3.29}$$

Two useful rules for operations on transition matrices are now stated. The composition rule shows how transition matrices may be combined,

$$\Phi(t_2,t_0) = \Phi(t_2,t_1)\,\Phi(t_1,t_0). \tag{3.30}$$

The inverse of a transition matrix, is functionally obtained by interchanging its arguments,

$$\Phi^{-1}(t_2,t_1) = \Phi(t_1,t_2). \tag{3.31}$$

Both rules are easily verified by application of the definition of the transition matrix given by equation (3.28).

The following relationship between the determinant of the transition matrix and the trace of the coefficient matrix A(t) is attributed to a combination of Abel, Jacobi, and Liouville [36]

$$\det \Phi(t,t_0) = \exp[\int_0^t tr\{A(\sigma)\}d\sigma] \tag{3.32}$$

For a Hamiltonian system it has been shown in (3.25) and (3.26) that the coefficient matrix A(t) from the variational equations is skew symplectic, hence the trace of the matrix A(t) is zero. From equation (3.32), the determinant of the transition matrix of a Hamiltonian system is one.

An important property of the transition matrix of a Hamiltonian system is that it is symplectic. As a consequence, the transition matrix, $\Phi(t,t_0)$, must satisfy equation (3.14) or an equivalent form, such as

$$K\Phi^T K^{-1} = \Phi^{-1}. \tag{3.33}$$

This equality may be verified by showing that $K\Phi^T K^{-1}$ is a solution to the adjoint system,

$$\dot{\Phi}^{-1}(t,t_0) = -\Phi^{-1}(t,t_0)A, \tag{3.34}$$

where A is the coefficient matrix from (3.25). The adjoint equations may be determined by operating on the identity

$$\Phi(t,t_0)\Phi^{-1}(t,t_o) = I. \tag{3.35}$$

Taking the time derivative and rearranging, results in the following relationship,

$$\dot{\Phi}(t,t_0)\Phi^{-1}(t,t_0) = -\Phi(t,t_0)\dot{\Phi}^{-1}(t,t_0). \tag{3.36}$$

Using equation (3.29) to eliminate $\Phi$ in the left-hand side of equation (3.36) and then premultiplying both sides of the equation by $-\Phi^{-1}$ gives the differential equation for the adjoint system previously identified in equation (3.34).

To verify equation (3.33), replace $\Phi^{-1}$ with $K\Phi^TK^{-1}$ in equation (3.34). This gives the following result,

$$K\dot{\Phi}^TK^{-1} = -K\Phi^TK^{-1}A. \tag{3.37}$$

Eliminating $\dot{\Phi}^T$ in equation (3.37) using equation (3.29) and postmultiplying both sides of the resulting equation by K gives

$$K\Phi^TA^T = -K\Phi^TK^{-1}AK. \tag{3.38}$$

Using the skew symplectic property of A given by equation (3.15), the above equation is reduced to an identity. This shows $K\Phi^TK^{-1}$ satisfies the adjoint equations.

Investigating initial conditions, when $t = t_0$, the following relationships of the transition matrix may be written,

$$\Phi(t_0,t_0) = \Phi^T(t_0,t_0) = \Phi^{-1}(t_0,t_0) = I, \tag{3.39}$$

$$K\Phi^T(t_0,t_0)K^{-1} = KIK^{-1} = I. \tag{3.40}$$

Since $K\Phi^TK^{-1}$ and $\Phi^{-1}$ satisfy the same adjoint equation (3.34) and they are equal at one point (3.40), they are equal everywhere. Hence

the equality of equation (3.33) is verified and the symplectic property of the transition matrix of a Hamiltonian system has been demonstrated.

### 3.3 Eigenvalues of the Transition Matrix

The basis of many of the properties of a linear system is its eigenvalues. They provide an intrinsic characterization of a matrix. Of principle interest are the relationships associated with a special matrix, the transition matrix. In this section some fundamental concepts are reviewed, a few useful relationships between matrices and their eigenvalues are presented, and finally an important property of the eigenvalues of a transition matrix is established.

Generally the eigenvalues of a matrix are introduced by reference to the following matrix relationship,

$$\Phi\zeta = \mu\zeta, \tag{3.41}$$

where $\Phi$ is a known (2n x 2n) matrix, $\zeta$ is a nonzero 2n vector, and $\mu$ is a constant scalar parameter. The particular values of $\mu$ that satisfy this relationship are the eigenvalues of $\Phi$. The vectors $\zeta$ corresponding to each of the eigenvalues are the eigenvectors. They are determined in direction only and form a set of basis vectors which define the 2n phase space of the system.

The matrix equation (3.41) may be rewritten as a set of 2n homogeneous equations,

$$[\Phi - \mu I]\zeta = 0. \tag{3.42}$$

Nontrivial solutions for $\zeta$ exist if, and only if, the inverse of the coefficient matrix $[\Phi-\mu I]$ does not exist [36]. This statement leads to the characteristic equation of $\Phi$ given by

$$\det\,[\Phi - \mu I] = 0. \tag{3.43}$$

This expression is a polynomial equation of order 2n in the scalar parameter $\mu$. Its 2n roots are the eigenvalues of $\Phi$ and may be real or complex.

When the elements of $\Phi$ are all real, it is easy to show that complex eigenvalues occur only in conjugate pairs. Taking the conjugate of elements of both sides of equation (3.41) results in

$$\Phi\bar{\zeta} = \bar{\mu}\bar{\zeta}, \tag{3.44}$$

where $\Phi$ is unchanged by the operation since its elements are real, and the bar above the remaining elements indicates their conjugate. Writing the characteristic equation of $\Phi$ from equation (3.44) gives

$$\det\,[\Phi - \bar{\mu} I] = 0. \tag{3.45}$$

This shows that the conjugate of a complex eigenvalue of a real matrix is also an eigenvalue of that matrix. As a result the complex eigenvalues of the transition matrix must occur in conjugate pairs.

Some useful relationships between any matrix and its eigenvalues may be obtained by comparing the characteristic equation of the matrix to its factored form in terms of its roots (eigenvalues),

$$\det\,[\Phi-\mu I] = (\mu_1-\mu)(\mu_2-\mu)\ \ldots\ (\mu_{2n}-\mu). \tag{3.46}$$

When $\mu$ is set equal to an eigenvalue, both sides of equation (3.46) are zero as expected. Letting $\mu = 0$ gives an expression for the determinant of the transition matrix,

$$\det \Phi = \mu_1 \mu_2 \cdots \mu_{2n}, \tag{3.47}$$

i.e., the determinant of a matrix equals the product of its eigenvalues. Expanding both sides of equation (3.46) and equating like coefficients of $\mu$ gives 2n relationships including (3.47). From the coefficient of $\mu$ to the 2n-1 power, it is determined that the trace of the matrix equals the sum of its eigenvalues,

$$\text{trace } \Phi = \sum_{i=1}^{2n} \phi_{ii} = \sum_{i=1}^{2n} \mu_i, \tag{3.48}$$

where $\phi_{ii}$ corresponds to the diagonal elements of $\Phi$.

The following three relationships are applicable to any transition matrix $\Phi$ and its eigenvalues. They lead to an important property of the transition matrix of a Hamiltonian system. The first relationship is that the eigenvalues of $\Phi^{-1}$ are the inverse of the eigenvalues of $\Phi$. This is shown by premultiplying equations (3.41) by $\Phi^{-1}$, dividing the result by $\mu$, and rearranging as follows.

$$\Phi\zeta = \mu\zeta, \tag{3.41}$$

$$\Phi^{-1}\phi\zeta = \mu\Phi^{-1}\zeta,$$

$$\Phi^{-1}\zeta = \frac{1}{\mu}\zeta . \tag{3.49}$$

The second relationship is that eigenvalues of $\Phi$ are invariant to a similarity transformation. The operation $C\Phi C^{-1}$, where C is a constant matrix, is a similarity transformation of $\Phi$. Premultiplying equation (3.41) by C, then inserting the identity expression, $C^{-1}C$, in the left-hand side of the result verifies this relationship as follows

$$\Phi\zeta = \mu\zeta, \tag{3.41}$$

$$C\Phi\zeta = \mu C\zeta,$$

$$C\Phi C^{-1}(C\zeta) = \mu(C\zeta). \tag{3.50}$$

The third relationship is that eigenvalues of $\Phi$ are invariant to the transpose. This may be shown by comparing the characteristic equation (3.43) for $\Phi$ with that for its transpose, $\Phi^T$. From the construction of the two equations, it is obvious, by interchanging row and column operations respectively that the resultant equations in terms of $\mu$ are identical. Hence the eigenvalues of $\Phi$ are also the eigenvalues of $\Phi^T$.

Applying the previous result (3.50) to this result and choosing the matrix C to be the fundamental symplectic matrix, K, gives

$$K\Phi^T K^{-1}(K\zeta) = \mu(K\zeta). \tag{3.52}$$

When $\Phi$ is the transition matrix of a Hamiltonian system, the above equation (3.52) reduces, using its symplectic property (3.32), to

$$\Phi^{-1}(K\zeta) = \mu(K\zeta). \tag{3.53}$$

This means that $\Phi^{-1}$ and $\Phi$ have the same eigenvalues. But the eigenvalues of $\Phi^{-1}$ were also shown to be the inverse of the eigenvalues of $\Phi$. This implies the very important result for a Hamiltonian system, namely that the eigenvalues of the transition matrix $\Phi$ and its inverse must occur in reciprocal pairs.

For physically realizable systems the elements of the transition matrix are real. This is derivable from the all real set of variational equations determined earlier by tacitly assuming a physically realizable system. Combining the property that complex eigenvalues of real valued matrices occur in conjugate pairs with the reciprocity property of eigenvalues of the transition matrix, strong restrictions on the complex eigenvalues of a transition matrix occur.

A single pair of complex eigenvalues are restricted to the unit circle in the complex plane. This is shown by first applying the reciprocity property to a pair of complex eigenvalues, where $j = \sqrt{-1}$,

$$\mu_1 = a + jb, \tag{3.54}$$

$$\mu_2 = \frac{1}{\mu_1} = \frac{a - jb}{a^2 + b^2} . \tag{3.55}$$

In order that $\mu_1$ and $\mu_2$ also be conjugates, the denominator in (3.55) must be one. Since the denominator is the square of the magnitude of either eigenvalue, it is apparent that the eigenvalues lie on the unit circle.

For more complicated systems, the complex eigenvalues may

occur off the unit circle in doubly coupled sets of four eigenvalues. Each set is composed of two pairs of complex conjugates, which may also be arranged as two pairs of reciprocal eigenvalues. Many other configurations are possible. The important point is that off the unit circle complex eigenvalues are restricted to occur in groups of at least four coupled eigenvalues. An example for four eigenvalues would be the relationships (3.54) and (3.55) plus the following two,

$$\mu_3 = a - jb, \text{ and} \tag{3.56}$$

$$\mu_4 = \frac{1}{\mu_3} = \frac{a + jb}{a^2 + b^2} . \tag{3.57}$$

3.4 Properties of the Monodromy Matrix

Many of the properties of Hamiltonian systems with periodic solutions involve the monodromy matrix. This special constant matrix maps the transition matrix forward one period in time. After deriving the relationships which define the monodromy matrix, three of its important properties will be derived to conclude this chapter.

Earlier it was shown by construction that the transition matrix satisfies the variational equation (3.29). Since the coefficient matrix, $A(t)$, is periodic of period $T$, then $\Phi(t+T,t_0)$ is a solution to the same linear differential equations as $\Phi(t,t_0)$. From equation (3.27), the two solutions must be related by a constant matrix, $\Gamma$, as

$$\Phi(t+T,t_0) = \Phi(t,t_0)\Gamma. \tag{3.58}$$

Evaluating equation (3.58) at the initial time defines the monodromy matrix $\Gamma$,

$$\Gamma \equiv \Phi(t_0+T,t_0). \tag{3.59}$$

The arguments will generally be dropped for the monodromy matrix, since it is a constant. It is important to point out that the monodromy matrix varies with changes in the initial time and for multiple periods. These relationships will be established in the two following properties.

The first property of the monodromy matrix to be presented here follows directly from its definition in equations (3.58) and (3.59). The relationship of the monodromy matrix to its associated transition matrix is determined by replacing t, in equation (3.58), by t+T, and using (3.58) again for the last relation,

$$\Phi(t+2T,t_0) = \Phi(t+T,t_0)\Gamma = \Phi(t,t_0)\Gamma^2. \tag{3.60}$$

Continuing this process gives the general formula,

$$\Phi(t+nT,t_0) = \Phi(t,t_0)\Gamma^n, \tag{3.61}$$

which defines the functional relationship for the mapping of the transition matrix over multiple periods by the monodromy matrix.

The second property shows that even though the monodromy matrix is dependent on the initial time, $t_0$, its eigenvalues are invariant to such changes. Using the composition rule, the relationship between the monodromy matrix initialized at some time $t_1$ to the one at

some other time $t_2$ may be written as

$$\Phi(t_1+T,t_1) = \Phi(t_1+T,t_2+T)\Phi(t_2+T,t_2)\Phi(t_2,t_1),$$

$$\Gamma(T,t_1) = \Phi(t_1+T,t_2+T)\Gamma(T,t_2)\Phi(t_2,t_1). \tag{3.62}$$

Using the composition rule and the matrix inverse rule for the first matrix on the right hand side of (3.62) gives

$$\Phi(t_1+T,t_2+T) = \Phi(t_1+T,t_0)[\Phi(t_2+T,t_0)]^{-1} . \tag{3.63}$$

From the definition of the monodromy matrix and the previous rules, the following result for (3.63) is obtained

$$\begin{aligned}\Phi(t_1+T,t_2+T) &= \Phi(t_1,t_0)\Gamma(T,t_0)[\Phi(t_2,t_0)\Gamma(T,t_0)]^{-1} ,\\ &= \Phi(t_1,t_0)\Gamma\Gamma^{-1}\Phi^{-1}(t_2,t_0),\\ &= \Phi(t_1,t_0)\Phi(t_0,t_2),\\ &= \Phi(t_1,t_2).\end{aligned} \tag{3.64}$$

Equation (3.62) can now be expressed, using (3.64) and the transition matrix inverse rule as

$$\Gamma(T,t_1) = \Phi^{-1}(t_2,t_1)\Gamma(T,t_2)\Phi(t_2,t_1). \tag{3.65}$$

This expression shows that the monodromy matrix for a Hamiltonian system initialized at one time is related to the monodromy matrix of the system initialized at another time by a similarity transformation. Thus from equation (3.50), the eigenvalues of the monodromy matrix are

invariant with respect to its initialization.

The third property shows that there are at least two unity eigenvalues of the monodromy matrix for any autonomous Hamiltonian system with periodic solutions. Using the system equations expressed in (3.9) and (3.10), rewritten below for convenience,

$$\dot{y} = Y(y), \tag{3.9}$$

$$y(t+T) = y(t), \tag{3.10}$$

the variational equations, also represented by equation (3.25), can be expressed as

$$\delta\dot{y} = \frac{\partial Y}{\partial y}\,\delta y \ . \tag{3.66}$$

Taking the time derivative of equation (3.9) gives

$$\frac{d\dot{y}}{dt} = \frac{\partial Y}{\partial y}\,\dot{y} \tag{3.67}$$

which shows that $\dot{y}(t)$ is a solution to the variational equations (3.66). From equation (3.27) it is apparent that a constant vector C exists such that

$$\dot{y}(t) = \Phi(t,t_0)C, \tag{3.68}$$

where $\Phi(t,t_0)$ is the transition matrix related to the variational equations (3.9). In other words, $\dot{y}(t)$ must be a linear combination of the columns of the transition matrix. At the initial time $t = t_0$, the transition matrix reduces to the identity matrix and the constant

matrix $C = \dot{y}(t_0)$. Then equation (3.68) may be rewritten

$$\dot{y}(t) = \Phi(t,t_0)\dot{y}(t_0). \tag{3.69}$$

After one period, T, equation (3.69) can be expressed using equations (3.9), (3.10), and (3.68) as

$$\dot{y}(t_0) = \Phi(t_0+T,t_0)\dot{y}(t_0) ,$$

$$\dot{y}(t_0) = \Gamma(T,t_0)\dot{y}(t_0). \tag{3.70}$$

Rearranging equation (3.70) in the form of equation (3.40) gives

$$[\Gamma - I]\dot{y}(t_0) = 0, \tag{3.71}$$

which shows that the monodromy matrix has one unity eigenvalue. The eigenvector corresponding to this unity, eigenvalue is the tangent vector to the periodic path of the solution. It is defined in the 2n-phase space of the system at the initial time. A second unity eigenvalue of the monodromy matrix must also exist due to the previously reviewed property that the eigenvalues of a Hamiltonian system must occur in reciprocal pairs. As was shown in the first section of this chapter and as is implied by the two unity eigenvalues, the order of the system is reducible by at least two.

## CHAPTER 4

## NEW THEORETICAL RESULTS

Contributions to the general theory of optimal periodic control developed in the course of this research are presented in this chapter. A principle result is derived in each of the following four sections. In the first section, a new algebraic Riccati equation is derived, which determines the initial conditions for generating the periodic solutions of the Riccati differential equation associated with the second variation of the performance index. An extremely useful similarity transformation of the monodromy matrix is developed from this result. In the second section, a large class of extrema for the optimal periodic control problem is shown to be non-optimizing. The symplectic properties of the monodromy matrix and the previous similarity transformation are used to obtain this result. A weak sufficiency condition from the second variation is developed for the free period case in the third section. Finally, a periodic regulator and its control law are determined by investigating optimal paths near a local periodic optimum.

### 4.1 An Algebraic Matrix Riccati Equation

A new algebraic equation is derived relating the Riccati variable and elements of the monodromy matrix. The variational equations, which determine the transition matrix for the state variables and Lagrange multipliers, are obtained from the Euler-Lagrange

equations, (2.9) through (2.11), by taking variations about an extremal solution. Retaining only the linear terms in the variations gives the following 2n + m equations,

$$\delta\dot{x} = f_x(t)\delta x + f_u(t)\delta u, \tag{4.1}$$

$$\delta\dot{\lambda} = -H_{xx}(t)\delta x - H_{xu}(t)\delta u - f_x^T(t)\delta\lambda, \tag{4.2}$$

$$0 = H_{ux}(t)\delta x + H_{uu}(t)\delta u + f_u^T(t)\delta\lambda, \tag{4.3}$$

where the partials have been evaluated along the extremal path.

Provided $H_{uu}$ is nonsingular, the m control variations, $\delta u$, may be written from (4.3) in terms of the n variations in the state, $\delta x$, and the n variations in the Lagrange multipliers, $\delta\lambda$, as

$$\delta u = -(H_{uu})^{-1}[H_{ux}\delta x + f_u^T\,\delta\lambda] . \tag{4.4}$$

Eliminating the control variations (4.4) from the 2n equations (4.1) and (4.2) determines the variational equations in the form of (3.25),

$$\begin{bmatrix} \delta\dot{x} \\ \delta\dot{\lambda} \end{bmatrix} = \begin{bmatrix} f_x - f_u(H_{uu})^{-1}H_{ux} & -f_u(H_{uu})^{-1}f_u^T \\ -H_{xx} + H_{xu}(H_{uu})^{-1}H_{ux} & -f_x^T + H_{xu}(H_{uu})^{-1}f_u^T \end{bmatrix} \begin{bmatrix} \delta x \\ \delta\lambda \end{bmatrix}, \tag{4.5}$$

where again the partial derivatives are evaluated along the extremal path. Just as in equation (3.25), the elements of the coefficient matrix of (4.5) are periodic functions of time with the same period, T, as the extremal solutions.

The transition matrix may be obtained by integrating equation (3.29) using the initial condition, $\Phi(t_0,t_0) = I$, and the coefficient matrix from equation (4.5). This results in

$$\Phi(t,t_0) = \begin{bmatrix} \Phi_{11}(t,t_0) & \Phi_{12}(t,t_0) \\ \Phi_{21}(t,t_0) & \Phi_{22}(t,t_0) \end{bmatrix}, \tag{4.6}$$

where the partitions are each (n x n) submatrices. This matrix maps perturbations in the state and Lagrange multipliers from some initial time, $t_0$, to a subsequent time, t, as follows,

$$\begin{bmatrix} \delta x(t) \\ \delta\lambda(t) \end{bmatrix} = \begin{bmatrix} \Phi_{11}(t,t_0) & \Phi_{12}(t,t_0) \\ \Phi_{21}(t,t_0) & \Phi_{22}(t,t_0) \end{bmatrix} \begin{bmatrix} \delta x(t_0) \\ \delta\lambda(t_0) \end{bmatrix}. \tag{4.7}$$

An expression relating a perturbation in the state to one in the Lagrange multiplier for any time can be determined by manipulating equation (4.7). Assume the following general relationship,

$$\delta\lambda(t) = P(t)\,\delta x(t), \tag{4.8}$$

where P(t) is an (n x n) matrix with elements, continuous functions of time. Using this relationship in equation (4.7) to eliminate $\delta\lambda(t_0)$ results in,

$$\begin{bmatrix} \delta x(t) \\ \delta\lambda(t) \end{bmatrix} = \begin{bmatrix} \Phi_{11}(t,t_0) + \Phi_{12}(t,t_0)P(t_0) \\ \Phi_{21}(t,t_0) + \Phi_{22}(t,t_0)P(t_0) \end{bmatrix} \delta x(t_0), \tag{4.9}$$

which for convenience may be abbreviated as

$$\begin{bmatrix} \delta x(t) \\ \delta\lambda(t) \end{bmatrix} = \begin{bmatrix} X(t) \\ \Lambda(t) \end{bmatrix} \delta x(t_0). \tag{4.10}$$

Provided $[X(t)]^{-1}$ exists, the variation of the state at the initial time, $\delta x(t_0)$, can be eliminated in equation (4.10) leaving the result

$$\delta\lambda(t) = \Lambda(t)[X(t)]^{-1}\delta x(t). \tag{4.11}$$

By comparing equations (4.8) and (4.11), the following expression for P(t) can be written,

$$P(t) = \Lambda(t)[X(t)]^{-1}, \tag{4.12}$$

which in expanded form is,

$$P(t) = [\Phi_{21}(t,t_0) + \Phi_{22}(t,t_0)P(t_0)] \cdot [\Phi_{11}(t,t_0) + \Phi_{12}(t,t_0)P(t_0)]^{-1}. \tag{4.13}$$

This expression for P(t) represents the solutions to the Riccati-type differential equation (2.15), rewritten below, which was derived from the second variation,

$$\dot{P}(t) = [-f_x^T + H_{xu}(H_{uu})^{-1}f_u^T]P(t) - P(t)[f_x - f_u(H_{uu})^{-1}H_{ux}] + P(t)[f_u(H_{uu})^{-1}f_u^T]P(t) - H_{xx} + H_{xu}(H_{uu})^{-1}H_{ux}. \tag{2.15}$$

The details of showing (4.13) satisfies (2.15) are presented in appendix A.

The transpose of P(t) in the second and third terms of equation (2.15) has been dropped since P(t) is necessarily symmetric. This is easily shown by taking the transpose of both sides of the equation.

The result is the same differential equation for $P^T(t)$ as for $P(t)$, which implies that $P(t)$ is symmetric if it was initially symmetric,

$$P^T(t) \quad = \quad P(t). \tag{4.14}$$

Now assume, that a periodic solution exists to the Riccati type equation (2.15) which has a period, T, the same as or an integral multiple of the period of the coefficients of P in the equation. An expression for the initial value matrix of the Riccati variable, $P(t_0)$, may be obtained by evaluating equation (4.13) at $t = T$ and using the periodicity condition $P(t_0+T) = P(t_0)$. This provides

$$\begin{aligned} P(t_0) \quad = \quad & [\Phi_{21}(t_0+T,t_0) + \Phi_{22}(t_0+T,t_0)P(t_0)] \\ & \cdot[\Phi_{11}(t_0+T,t_0) + \Phi_{12}(t_0+T,t_0)P(t_0)]^{-1}. \end{aligned} \tag{4.15}$$

Using the relationship (3.56), between the monodromy matrix and the transition matrix, in the above expression and then rearranging terms results in the following algebraic Riccati equation

$$P(t_0)\Gamma_{12}P(t_0) + P(t_0)\Gamma_{11} - \Gamma_{22}P(t_0) - \Gamma_{21} \quad = \quad 0. \tag{4.16}$$

Solutions to this equation provide the initial conditions, $P(t_0)$, that generate all of the periodic solutions to the Riccati differential equation (2.15).

Since the initial time, $t_0$, is arbitrary, the algebraic expression (4.16) is valid for any time. The time relationships for the coefficients of P are determined using the similarity transformation (3.63) when the monodromy matrix is known for some initial

time, $t_1$, and period T. The coefficients are then the appropriate blocks of the resultant, time-varying monodromy matrix,

$$\Gamma(T,t) = \Phi(t,t_1)\Gamma(T,t_1)\Phi^{-1}(t,t_1), \tag{4.17}$$

where the time $t_0$ has been replaced by t to emphasize the variability in time. In this manner, the Riccati variable may be determined for any time using a given monodromy matrix and the transition matrix.

Another useful result of equation (4.16) is derived from a similarity transformation of the monodromy matrix. An off-diagonal block of the transformed matrix, partitioned into four (n x n) blocks, is reduced identically to zero. The relationships and properties derived from this simple operation are so important to the new developments in this chapter that the transformation is presented here as a theorem.

THEOREM 4.1: If a periodic solution to the matrix Riccati differential equation (2.15) exists with period T (the same or an integral multiple of the period of the coefficients of P in the equation), then the monodromy matrix, given by

$$\Gamma(T,t_0) = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix}, \tag{4.18}$$

is similar to the transformed matrix

$$L\Gamma(T,t_0)L^{-1} = \begin{bmatrix} \Gamma_{11} + \Gamma_{12}P(t_0) & \Gamma_{12} \\ 0 & \Gamma_{22}-P(t_0)\Gamma_{12} \end{bmatrix}, \tag{4.19}$$

where the similarity transformation matrix, L, is defined by

$$L \equiv \begin{bmatrix} I & 0 \\ -P(t_0) & I \end{bmatrix}, \qquad (4.20)$$

and $P(t_0)$ is the initial value matrix that satisfied the algebraic Riccati equation (4.16).

PROOF: Perform the matrix operations of the similarity transformation indicated by the left-hand side of (4.19) where the inverse of L is

$$L^{-1} = \begin{bmatrix} I & 0 \\ P(t_0) & I \end{bmatrix}. \qquad (4.21)$$

This results in a matrix with the three non-zero blocks of (4.19) and a term which is the algebraic Riccati equation (4.16) in the remaining block. Due to the choice of $P(t_0)$, this block is identically zero and the transformed matrix is the desired one.

## 4.2 Eigenvalues and Sufficient Conditions

It was shown in section 3.2 that the monodromy matrix of the Hamiltonian system defined by the periodic control problem, (1.1) through (1.3), is symplectic. The resulting properties of the monodromy matrix and those associated with the transformed matrix of theorem 4.1 from the previous section, restrict the eigenvalues of the monodromy matrix for optimal solutions. It is shown that, except for

possibly isolated limit points, local optimal solutions are all unstable. The sufficient conditions [14] for a local minimum, presented in section 2.2, are thus modified by the developments in this section.

Since the monodromy matrix is symplectic, it must satisfy the relationship (3.14) resulting in the following equation,

$$\Gamma K \Gamma^T = K, \tag{4.22}$$

where K is the fundamental symplectic matrix (3.7). Writing this relationship in block form, consistent with the partitioning of equation (4.18), and using equations (3.17) through (3.19) gives

$$\Gamma_{11}\Gamma_{12}^T = \Gamma_{12}\Gamma_{11}^T , \tag{4.23}$$

$$\Gamma_{22}\Gamma_{21}^T = \Gamma_{21}\Gamma_{22}^T , \tag{4.24}$$

$$\Gamma_{11}\Gamma_{22}^T - \Gamma_{12}\Gamma_{21}^T = I. \tag{4.25}$$

The symplectic property of the monodromy matrix is preserved through the similarity transformation in the theorem of the preceding section. This may be demonstrated by showing that the transformed matrix $L\Gamma L^{-1}$ satisfies the following relationship derived from the symplectic property (3.14),

$$(L\Gamma L^{-1})K(L\Gamma L^{-1})^T = K. \tag{4.26}$$

Premultiplying both sides of (4.26) by $L^{-1}$ and postmultiplying by $L^{-T}$ gives

$$\Gamma(L^{-1}KL^{-T})\Gamma^T = (L^{-1}KL^{-T}). \tag{4.27}$$

If the matrix L is symplectic, then from (3.14) the following equivalent relationships hold,

$$LKL^T = K \quad \text{and} \quad K = L^{-1}KL^{-T}, \tag{4.28}$$

and equation (4.27) reduces to the identity given by (4.22). Performing the matrix operations indicated by either of equations (4.28) verifies that both matrices L and $L\Gamma L^{-1}$ are symplectic.

Expressing $L\Gamma L^{-1}$ in the block form (4.19) and expanding equation (4.26) to obtain relationships between combinations of elements of the resulting symplectic matrix, as in (3.17) through (3.19), provides only one new, but very important equation,

$$[\Gamma_{11} + \Gamma_{12}P][\Gamma_{22} - P\Gamma_{12}]^T = I, \text{ or}$$

$$[\Gamma_{22} - P\Gamma_{12}]^T = [\Gamma_{11} + \Gamma_{12}P]^{-1} . \tag{4.29}$$

The significance of this new equation is that it strongly restricts the eigenvalues of the monodromy matrix corresponding to real-valued Riccati variable elements. Note that the eigenvalues of the two matrices, $[\Gamma_{11} + \Gamma_{12}P]$ and $[\Gamma_{22} - P\Gamma_{12}]$ from equations (4.19) and (4.29), are also the eigenvalues of the monodromy matrix, $\Gamma$, since eigenvalues are invariant through a similarity transformation, as established previously by (3.50). The transformation also assigns the eigenvalues of the monodromy matrix to the two submatrices on the diagonal of the transformed matrix such that the eigenvalues of one are the reciprocal of those of the other submatrix.

By considering small perturbations about the extremal solution, the restrictions on the eigenvalues due to equation (4.29) become apparent. The monodromy matrix maps perturbations over one complete cycle or period. The elements of this matrix (a special case of the transition matrix) are always real-valued, provided the application is physically realizable. Also, the elements of the associated Riccati matrix must be real-valued for the extremal solutions to be locally optimizing. As a result each element of the two matrices of (4.29) must be real-valued, requiring, from (3.44) and (3.45), that the complex eigenvalues of each matrix must occur in conjugate pairs. The reciprocal relationship (4.29) between the two matrices requires that complex eigenvalues of the monodromy matrix, on the unit circle, must occur in even multiple conjugate pairs, i.e., double conjugate pairs, quadruple conjugate pairs, and so forth. Generally, this special circumstance results at a limit point where coupled eigenvalues coalesce upon entering or exiting the unit circle.

Periodic solutions are commonly classified as stable when all eigenvalues of the monodromy matrix lie on the unit circle and unstable when at least one pair lies off the unit circle. Consequently, locally optimizing periodic solutions are unstable with the possible exception of isolated limit points. This statement further restricts the sufficient conditions [14] for a local optimum presented in section 2.2.

A deficiency in the original statement of these conditions is now examined. Recall from section 3.4 that the monodromy matrix for a periodic Hamiltionian system always has at least two unity eigenvalues.

Due to the relationship (4.29), at least one unity eigenvalue will occur in each of the submatrices on the diagonal of the transformed matrix (4.19). Since Z of equation (2.17) is identically equal to $[\Gamma_{11} + \Gamma_{12}P]$, one of the submatrices of (4.19), the condition (5) of the sufficiency conditions is never satisfied.

From equation (3.71) the eigenvector corresponding to one of the unity eigenvalues, is tangent to the reference (extremal) solution in phase space. As a result, small perturbations along the tangent path do not change the amplitude or period of the perturbed solution. The only apparent effect when comparing the perturbed solution to the unperturbed solution is a time or phase shift. If, as in the general problem (1.1) through (1.3), the only condition is the periodicity constraint, the optimal periodic solution is invariant to a perturbation along its tangent.

## 4.3 Free Period Second Variation Condition

The weak sufficient conditions for optimality derived by Bittanti, Locatelli, and Maffezzoni [14] for periodic control problems with fixed period are extended in this section to the class of problems for which the period is unrestricted. The modified conditions, two of which are also applicable to the fixed period case, are developed from the second variation applying results obtained from the Hamilton-Jacobi theory.

The second variation may be expressed in terms of operations on the first variation, given by equation (2.8) as follows,

$$d^2J = d(dJ) = d\{dv^T\Psi + (v^T - \tfrac{1}{T}\lambda^T)dx\Big|_{t=0}^{t=T}\} + d\{\tfrac{1}{T}(H-J)dT\}$$

$$+\ d\{\ \frac{1}{T}\int_0^T[(H_x + \dot{\lambda}^T)\delta x + H_u\delta u + \delta\lambda^T(H_\lambda^T - \dot{x})]dt\}. \quad (4.30)$$

The first variation conditions, (2.9) through (2.14), must be satisfied as well as the following variational constraints,

$$\delta\dot{x} \ = \ f_x\delta x \ + \ f_u\delta u, \quad (4.31)$$

$$dx(T) \ = \ dx(0), \quad (4.32)$$

which are the variations of the system constraints, (1.2) and (1.3), to the first order.

The expression for the second variation (4.30) may be reduced in the following manner. Using equations (1.3), (2.13), and (4.32) in the first term of the expression gives,

$$d\{dv^T\Psi + (v^T - \tfrac{1}{T}\lambda^T)dx\Big|_{t=0}^{t=T}\} = (2dv^T + \tfrac{1}{T^2}\lambda^T dT - \tfrac{1}{T}\ d\lambda^T)dx\ \Big|_{t=0,}^{t=T}$$

$$= -\ \frac{1}{T}d\lambda^T dx\ \Big|_{t=0}^{t=T}\ . \quad (4.33)$$

Using equations (2.11), (2.14), and dJ = 0 in the second term of the expression (4.30) gives

$$d\{\tfrac{1}{T}(H-J)dT\} \ = \ -\frac{1}{T^2}(H-J)dT^2 + \frac{1}{T}\ (dH-dJ)dT,$$

$$= \ \frac{1}{T}(H_x dx + d\lambda^T H_\lambda^T)\ \Big|^{t=T}\ dT\ . \quad (4.34)$$

In the last term of expression (4.30), using conditions (2.9) through (2.11), equation (4.31), and integrating by parts the $\delta\dot{\lambda}^T$ term gives

$$d\{\frac{1}{T}\int_0^T [EL]dt\} = -\frac{1}{T^2}dT\int_0^T [EL]dt + \frac{1}{T}[EL]dt\Big|_{t=0}^{t=T} + \frac{1}{T}\int_0^T \delta[EL]dt,$$

$$= \frac{1}{T}(\delta\lambda^T\delta x)\Big|_{t=0}^{t=T} + \frac{1}{T}\int_0^T [\delta x^T \delta u^T]\begin{bmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu}\end{bmatrix}\begin{bmatrix}\delta x \\ \delta u\end{bmatrix} dt, \qquad (4.35)$$

where $[EL] = [(H_x+\dot{\lambda}^T)\delta x + H_u\delta u + \delta\lambda^T(H_\lambda^T-\dot{x})]$.

Combining these results, equations (4.33) through (4.35), for the second variation provides the following expression,

$$d^2J = \frac{1}{T}[(\delta\lambda^T\delta x - d\lambda^T dx)\Big|_{t=0}^{t=T} + (H_x dx + d\lambda^T H_\lambda^T)\Big|^{t=T} dT]$$

$$+ \frac{1}{T}\int_0^T [\delta x^T \delta u^T]\begin{bmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu}\end{bmatrix}\begin{bmatrix}\delta x \\ \delta u\end{bmatrix} dt. \qquad (4.36)$$

Replacing the $\delta\lambda^T\delta x$ term using the relationship (1.4) and conditions (2.9) through (2.11) gives a more useful form of the second variation,

$$d^2J = \frac{1}{T}[dx^T(T)dT]\begin{bmatrix} 0 & H_x^T(T) \\ H_x(T) & -H_x(T)f(T)\end{bmatrix}\begin{bmatrix} dx(T) \\ dT\end{bmatrix}$$

$$+ \frac{1}{T}\int_0^T [\delta x^T \delta u^T]\begin{bmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu}\end{bmatrix}\begin{bmatrix}\delta x \\ \delta u\end{bmatrix} dt, \qquad (4.37)$$

subject to the variational constraints given by (4.31) and (4.32).

As in the accessory minimum problem [15], showing that

$d^2J > 0$ for any $\delta u(t) \neq 0$ establishes the minimality of the extremal solution. This may be accomplished by adding to equation (4.37) the identically zero quantity,

$$\frac{2}{T}\int_0^T \delta x^T P(f_x\delta x + f_u\delta u - \delta\dot{x})dt \;=\; 0, \qquad (4.38)$$

where the partial derivatives of f and H are evaluated along the extremal trajectory, and P is the Riccati variable determined by the matrix differential equation (4.14). The integral of the resulting relationship for the second variation can be written as a perfect square after integrating the $\delta\dot{x}$ term by parts and using equation (4.14) to appropriately express the coefficients of the variations. This gives for the second variation

$$d^2J = \frac{1}{T}\left\{2H_x(T)dx(T)dT - H_x(T)f(T)dT^2 - (\delta x^T(t)P(t)\delta x(t))\Big|_{t=0}^{t=T}\right\}$$
$$+\frac{1}{T}\int_0^T [\delta x^T \;\; \delta u^T]\begin{bmatrix}(H_{xu}+Pf_u)(H_{uu})^{-1}f_u^TP+H_{ux}) & (H_{ux}+Pf_u) \\ (f_u^TP+H_{ux}) & H_{uu}\end{bmatrix}\begin{bmatrix}\delta x \\ \delta u\end{bmatrix} dt, \qquad (4.39)$$

which is commonly expressed in the normed form as,

$$d^2J = \frac{1}{T}\left\{2H_x(T)dx(T)dT - H_x(T)f(T)dT^2 - (\delta x^T(t)P(t)\delta x(t))\Big|_{t=0}^{t=T}\right\}$$
$$+\frac{1}{T}\int_0^T \left\|(H_{uu})^{-1}(f_u^T P + H_{ux})\delta x + \delta u\right\|^2_{H_{uu}} dt. \qquad (4.40)$$

The end point expression in (4.40) will be shown to vanish. First the following relationship must be developed.

$$\dot{\lambda}(0) \;\equiv\; \dot{\lambda}(0) - P(0)\,\dot{x}\,(0) \;=\; 0. \qquad (4.41)$$

From equation (3.71), the relationship between the monodromy matrix, a unity eigenvalue, and the associated eigenvector may be expressed as

$$\Gamma\dot{y}(0) \quad = \quad \dot{y}(0), \tag{4.42}$$

where the eigenvector y(0) is the phase space velocity vector given by

$$\dot{y}(0) \quad = \quad \begin{bmatrix} \dot{x}(0) \\ \dot{\lambda}(0) \end{bmatrix}, \tag{4.43}$$

Performing a transformation of variables in (4.42) using the similarity transformation matrix, L, defined by (4.20) provides the following relationship,

$$[L\Gamma L^{-1}]L\dot{y}(0) \quad = \quad L\dot{y}(0) \ . \tag{4.44}$$

By substituting in (4.44) the equivalent expressions given by (4.19), (4.41), and (4.43), the following matrix equation is obtained,

$$\begin{bmatrix} \Gamma_{11} + \Gamma_{12}P & \Gamma_{12} \\ 0 & \Gamma_{22} - P\Gamma_{12} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\bar{\lambda}} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{\bar{\lambda}} \end{bmatrix}. \tag{4.45}$$

Assume first that $\dot{\bar{\lambda}} \neq 0$ and satisfies the bottom equation of (4.45), which may be expressed as

$$[\lambda_{22} - P\lambda_{12} - I]\ \dot{\bar{\lambda}} \quad = \quad 0 \ . \tag{4.46}$$

Premultiplying the upper equation of (4.45) by $\dot{\bar{\lambda}}^T$ gives the result,

$$\dot{\bar{\lambda}}^T(\Gamma_{11} + \Gamma_{12}P - I)\dot{x} + \dot{\bar{\lambda}}^T\Gamma_{12}\dot{\bar{\lambda}} = 0. \tag{4.47}$$

The first term in this expression is reduced to zero by the following operations. First, multiply equation (4.46) by $(\Gamma_{22} - P\Gamma_{12})^{-1}$, which gives

$$[I - (\Gamma_{22} - P\Gamma_{12})^{-1}]\dot{\bar{\lambda}} = 0. \tag{4.48}$$

Using the symplectic property derived in (4.29) for the transformed monodromy matrix provides

$$[I - (\Gamma_{11} + \Gamma_{12}P)^T]\dot{\bar{\lambda}} = 0, \tag{4.49}$$

which after transposing becomes

$$\dot{\bar{\lambda}}^T(\Gamma_{11} + \Gamma_{12}P - I) = 0. \tag{4.50}$$

Using this expression reduced equation (4.47) to

$$\dot{\bar{\lambda}}^T\Gamma_{12}\dot{\bar{\lambda}} = 0. \tag{4.51}$$

This equation may be written in terms of the symmetric part of $\Gamma_{12}$,

$$\dot{\bar{\lambda}}^T\left(\frac{\Gamma_{12} + \Gamma_{12}^T}{2}\right)\dot{\bar{\lambda}} = 0, \tag{4.52}$$

since the result is a scalar. However, in general

$$\left(\frac{\Gamma_{12} + \Gamma_{12}^T}{2}\right)\dot{\bar{\lambda}} \neq 0 \tag{4.53}$$

for $\dot{\bar{\lambda}} \neq 0$, obtained from equation (4.46), which contradicts the

original supposition; thus

$$\dot{\bar{\lambda}} = 0, \tag{4.54}$$

justifying the relationship (4.41).

Using the relationship for the variations (1.4), the relationships (2.9) through (2.10), and the result (4.41); the end point expression in (4.40) is identically zero, as shown below,

$$\begin{aligned} 2H_x^o dxdT - H_x^o f^o dT^2 \Big|_{t=T} &= \delta x^T P \delta x \Big|_{t=T} - \delta x^T P \delta x \Big|_{t=0}, \\ &= -(\dot{x}^T P dx + dx^T P \dot{x} - \dot{x}^T P \dot{x}\ dT)dT \Big|_{t=T} \\ &= 2H_x^o dxdT - H_x^o f^o dT^2. \end{aligned} \tag{4.55}$$

This leaves for the second variation the squared expression,

$$d^2 J = \frac{1}{T}\int_0^T \left\| (H_{uu}^o)^{-1}(f_u^{oT} P + H_{ux}^o)\delta x + \delta u \right\|^2_{H_{uu}^o} dt. \tag{4.56}$$

The neighboring optimal control law that causes the second variation (4.56) to vanish is

$$\delta u^o = -(H_{uu}^o)^{-1}(f_u^o\ P + H_{ux}^o)\ \delta x. \tag{4.57}$$

Any other control variation produces a greater cost since from equation (4.56), $d^2 J \geqq 0$.

In order to show that the extremal solution is locally minimizing, it must be shown that there is no neighboring periodic solution for which the generating variations satisfy the boundary condition

(4.32) and use the control law (4.57). The boundary condition may be rewritten using equation (1.4) in (4.32) and recognizing that $f^o$ is periodic with period T,

$$\delta x(0) = \delta x(T) + f^o(0)dT. \tag{4.58}$$

Using the neighboring optimal control law (4.57) in the dynamic constraint equation (4.31) provides the following linear, homogeneous differential equation for propagating the variation in the state,

$$\delta \dot{x} = [f_x^o - f_u^o(H_{uu}^o)^{-1}(f_u^{oT}P + H_{ux}^o)]\ \delta x. \tag{4.59}$$

This equation, (4.59), is the same as the first of the set of equations (4.5) with $\delta\lambda$ replaced by (4.8). The corresponding transition matrix from (4.9) may be expressed as

$$\Phi_Z = \Phi_{11} + \Phi_{12}P, \tag{4.60}$$

where Z is the coefficient matrix of (4.59), also defined in (2.17). It was shown in section 4.2 that the transition matrix $\Phi_Z$, evaluated over one period and expressed in terms of elements of the monodromy matrix, is

$$\Phi_Z(T,0) = \Gamma_{11} + \Gamma_{12}P(0), \tag{4.67}$$

and has at least one unity eigenvalue. Relative to this eigenvalue is the eigenvector $f^o(0)$, a vector tangent to the optimal periodic path at the initial time. This is the final relationship developed at the end of chapter three.

The class of all possible neighboring solutions is considered by examining the initial variation, $\delta x(0)$, that generates them. A variation parallel to the eigenvector $f^{o}(0)$ and an other one orthogonal to it are examined. Of concern is whether or not a set of variations can be found for which the periodicity constraint (4.58), is satisfied while the control law, (4.57), through the transition matrix (4.61) is applied. Any initial variation, $\delta x(0)$, can be represented by a combination of these two variations.

The relationship between the initial variation and the variation corresponding to the extremal (reference) trajectory after one period is given by

$$\delta x(T) = \Phi_Z(T,0)\delta x(0). \tag{4.62}$$

When the initial variation, $\delta x(0)$, is parallel to $f^{o}(0)$, the relationship (4.62) reduces to

$$\delta x(T) = \delta x(0). \tag{4.63}$$

Applying this result to the constraint (4.58) implies that the period is unchanged since the initial time is arbitrary, and a non-zero $f^{o}(0)$ can always be chosen. The variation at the initial time represents a change from one point on the extremal path to another point on the extremal path. Consequently, the component of a variation parallel to $f^{o}(0)$ produces no change in the optimal trajectory.

When the initial variation is orthogonal to $f^{o}(0)$, it is found, by applying the relationship (3.42) to each component of $\delta x(0)$

relative to the eigenvectors (basis vectors) of $\Phi_Z$, that $\delta x(T)$ is also orthogonal to $f^o(0)$. Provided the eigenvalues, excluding the unity eigenvalue corresponding to $f^o(0)$, do not lie on the unit circle, the following relationship is true for any variation orthogonal to $f^o(0)$,

$$\delta x(T) \neq \delta x(0). \tag{4.64}$$

However, because neither variation has a component in the direction of $f^o(0)$, there is a contradiction between (4.58) and (4.64). Therefore, with the conditions on the eigenvalues of $\Phi_Z$ stated above, no neighboring solution can be found which satisfies (4.58) and uses the control law (4.57). As a result, the extremal solution is a local minimum.

The sufficient conditions of Bittanti, Locatelli, and Maffezzoni [14] for local optima of periodic control problems as modified and developed in this section are summarized in the following theorem for either specified or unrestricted period:

THEOREM 4.2. For the periodic control problem defined by equations (1.1), (1.2), and (1.3), the control $u^o(\cdot)$, a piecewise-continuous, periodic function of the input space, is a local minimum if the following conditions are satisfied:

(1) The Euler-Lagrange equations, (2.9) through (2.11), and the transversality condition (2.13) are satisfied, where $u^o(\cdot)$ is a solution to (2.11) for $0 \leqq t \leqq T$ and $x^o(\cdot)$ and $\lambda^o(\cdot)$ are the corresponding solutions to the two-point boundary value problem which results from introducing $u^o(\cdot)$ into (2.9) and (2.10). When the

period is unrestricted, the additional transversality condition (2.14) must also be satisfied;

(2) The strong form of the Legendre condition is satisfied, $H^o_{uu} > 0$, for $0 \leqq t \leqq T$;

(3) A bounded symmetric solution to the Riccati equation (2.15), exists for $0 \leqq t \leqq T$ subject to the periodic boundary condition, (2.16); and

(4) Except for the unity eigenvalue associated with the eigenvector $f^o(0)$, no eigenvalue of $\Phi_Z(T,0)$ lies on the unit circle.

## 4.4 A Periodic Regulator

In the previous sections, an open-loop solution to the optimal periodic control problem (1.1), (1.2), and (1.3) was developed. Examining the restrictions on the eigenvalues of the monodromy matrix, specified in section 4.2, it is apparent that except for very special circumstances, the optimal open-loop solution is unstable. In this section a control strategy, analogous to the static regulator, is presented which minimizes the cost of holding a perturbed periodic system near its optimal open-loop trajectory.

Perturbations of a local optimal trajectory may be expressed in terms of the variational equations (4.5). Using the transformation matrix (4.21) of Theorem 4.1, a new set of variations may be defined as

$$\begin{bmatrix} \delta x(t) \\ \delta\lambda^*(t) \end{bmatrix} \equiv \begin{bmatrix} I & 0 \\ -P(t) & I \end{bmatrix} \begin{bmatrix} \delta x(t) \\ \delta\lambda(t) \end{bmatrix} , \qquad (4.65)$$

where P(t) is a solution of the Riccati equation (4.14). In terms of these transformed variables, the variational equations become

$$\begin{bmatrix} \delta\dot{x}(t) \\ \delta\dot{\lambda}^*(t) \end{bmatrix} = \begin{bmatrix} Z & -f_u^\circ (H_{uu}^\circ)^{-1} f_u^{\circ T} \\ 0 & -Z^T \end{bmatrix} \begin{bmatrix} \delta x(t) \\ \delta\lambda^*(t) \end{bmatrix}, \tag{4.66}$$

where Z is defined by equation (2.17) and the zero element is an identity expression of the Riccati equation (4.14). Note that the transformed variation, $\delta\lambda^*(t)$, is identically zero due to equation (4.8) and the relationship between variations of the state and co-state (Lagrange multipliers). Therefore equation (4.66) reduces to

$$\delta\dot{x}(t) = Z\,\delta x(t), \tag{4.67}$$

where the boundary condition $\delta x(0)$ is arbitrary. This equation propagates a perturbation to an optimal trajectory using the control law (4.57), which was incorporated into the variational equations through the transformation (4.65).

For the linear regulator problem, as in Bryson and Ho [15], the coefficients in the variational equations (4.5) are constants. This is the case for the static equilibrium solutions associated with the periodic control problem. In addition, the Riccati variable in (4.65) is a steady state solution of the Riccati equation (4.14). Therefore Z in (4.55) is also a constant. Provided the system is controllable, then, as in Brockett [36], a solution to $\dot{P} = 0$ exists such that the real parts of the eigenvalues of Z are negative and $\delta x(t) \to 0$ as $t \to \infty$. Using this P in the control law, equation (4.59), provides

the relationship for the static regulator of a linear system,

$$\delta u^o = -(H^o_{uu})^{-1} f_u^{oT} P \delta x. \tag{4.68}$$

Consider now a regulator for an optimal periodic trajectory (orbit). As indicated in section 4.1, the coefficients of the variational equation (4.5) are periodic. The relationship, (4.7), for the transition matrix shows how a perturbation will propagate away from an optimal orbit in the open loop control no feedback case. Provided a periodic solution to the Riccati equation (4.14) exists, Theorem 4.1 shows that in terms of the transformed variables (4.65) the propagation equations (4.7), evaluated over one period, T, becomes

$$\begin{bmatrix} \delta x(T) \\ \delta\lambda^*(T) \end{bmatrix} = \begin{bmatrix} \Gamma_{11} + \Gamma_{12}P & \Gamma_{12} \\ 0 & \Gamma_{22} - P\Gamma_{12} \end{bmatrix} \begin{bmatrix} \delta x(0) \\ \delta\lambda^*(0) \end{bmatrix}. \tag{4.69}$$

The initial variation, $\delta x(0)$, for (4.57) is arbitrary, and as before the transformed variation, $\delta\lambda^*(t)$, is identically zero. Therefore, the discrete equation (4.69) may be reduced for multiple stages to

$$\begin{aligned} \delta x(nT) &= [\Gamma_{11} + \Gamma_{12}P]\ \delta x((n-1)T), \\ &= \Phi_Z\ \delta x((n-1)T), \\ &= \left[\Phi_Z\right]^n \delta x\ (0), \end{aligned} \tag{4.70}$$

since the matrix $[\Gamma_{11} + \Gamma_{12}P]$ is a constant from stage to stage, its components having been evaluated at the time of the perturbation. As in equation (4.67) the optimal control law (4.57) is incorporated in

the discrete equation (4.70) which propagates perturbations to the optimal orbit.

In the closed loop (4.58) the optimal orbit forms a natural limit cycle for state perturbations provided that the eigenvalues of $\Phi_Z$ satisfy Theorem 4.2, condition (5). The components $\Gamma_{11}$ and $\Gamma_{12}$ of $\Phi_Z$ in (4.70) are both elements of the monodromy matrix of the optimal orbit and are uniquely determined by the initial time associated with the arbitrary state perturbation, $\delta x(0)$. The remaining component, P, is one of the solutions to the new algebraic Riccati equation (4.17). Each solution represents an initial condition which produces a periodic solution to the Riccati differential equation (4.14). As a result there is a one to one correspondence between solutions, P, to the algebraic Riccati equation and permissible combinations of the eigenvalues of the monodromy matrix, which comprise the eigenvalues of $\Phi_Z$. Therefore, the periodic regu- ator is the control law (4.57) determined by the particular solution, P, to (4.17) for which the eigenvalues of $\Phi_Z$ satisfy Theorem 4.2, condition (5). In some respects this is a generalization of the linear regulator problem.

A concept for implementing the periodic regulator may be developed which exploits two properties of the regulator. First, the gains for the regulator are dependent on parameters of the reference (optimal) orbit only. The time of the perturbation may be identified by a point on the orbit. Therefore, the gains may be computed apriori relative to states on the reference orbit and stored in a look-up

table. Second, the closed loop periodic system is invariant to perturbations tangent to the reference orbit. As a result, from an orthogonal projection of the perturbed state onto the reference orbit, both the new perturbation and the corresponding control gains can be determined for any desired interval along the perturbed path.

# CHAPTER 5

# AN OPTIMAL PERIODIC CONTROL PROBLEM

A particular optimal periodic control problem is formulated in this chapter and some preceding results are simplified by exploiting the order and symmetry of the problem. The dynamics are chosen to be simple, consisting of a double integration of the scalar control variable. The performance index, which is not convex in the state variables, is constructed to allow optimal periodic solutions for a range of control variable weighting. The principal objective of this chapter is to develop an example problem for which a local optimal periodic solution may be obtained that illustrates some of the general characteristics of the class of optimal periodic control problems. Recent efforts are briefly reviewed in the first section to provide the background from which the present research effort is motivated. In the next section the example problem is formulated, the first order necessary conditions are examined, and the frequency test is applied. Some simplifications of results presented in previous chapters are developed in section three for fourth order, symmetric systems such as the example problem. Finally, in the last section, an explicit solution to the Riccati equation is derived for any symmetric, fourth order, periodic system.

## 5.1 Background

An interesting application of optimal periodic control is

developed in a lengthy controversy [5, 6, 7, 8] over a particular aerospace problem. The contention pertains to the optimality with respect to fuel consumption of the steady-state cruise of an aircraft. The results obtained by Speyer [5] stimulated additional work [9], attempting to find an optimal periodic solution for the cruise segment of an aircraft flight path. Some of the important aspects of this aerospace problem are now briefly reviewed to establish the background and rationale for constructing a new and simpler optimal periodic control problem. The fabricated problem is thoroughly examined in the remainder of this work.

Steady-state cruise is established for an aircraft by minimizing its range factor, defined as fuel rate per range rate. All control variables are held constant when operating at this static cruise point. A frequently used dynamic model for evaluating aircraft performance is the energy state model developed by Bryson, Desai, and Hoffman [37], where altitude and thrust are control variables. For many aircraft, this energy state space produces a hodograph which is not convex. The physical explanation for this is that while maintaining a constant energy, there is one altitude where the aircraft flies aerodynamically efficient and another altitude where it flies thrust (or power) efficient. This dichotomy leads to the chattering (or relaxed) controls solution derived by Gilbert [35].

To eliminate the non-convexity in the hodograph and preserve an "optimal" steady-state cruise, Schultz and Zagalsky [6] revised the energy state model so that altitude becomes a state variable while flight path angle and thrust form the control space. For this model

the first order conditions are satisfied using controls lying in a region interior to the control space where the Legendre-Clebsch condition is met only in weak form. This is the result of the Hamiltonian being a linear function of both control variables and is called a doubly, singular arc. However, Speyer [7] shows that the generalized Legendre-Clebsch (or Kelley) condition [38] is not satisfied for this case, and therefore, the steady-state cruise arc is non-optimal.

Schultz [8] once more revised the model; this time so the flight path angle becomes a state variable, and the angle-of-attack and thrust form the control space. Now with this model, the Kelley condition is satisfied. Hence, the performance index with respect to the controls in the vicinity of the steady-state cruise point appears to be convex. However, once again Speyer [5] shows that the static aircraft cruise is non-minimizing. Since the cruise arc in question is time invariant, a transformation to the frequency domain can be made which greatly simplifies the Jacobi test, the final and most difficult condition to apply. Assuming constant mass, infinite time, and fixed end-points, the frequency test [5, 11, 18] shows that cyclic solutions produce better performance than the steady-state solution over the same period. However, the test does not give any indication of what the optimal path might be and, as yet, this remains an interesting and unsolved problem [39].

Continuing Speyer's work, Walker [9] searched for an optimal periodic flight path. However, the standard numerical optimization techniques he uses, which include steepest ascent and conjugate

gradient methods, do not converge to a solution. During the initial research for this dissertation, a similar effort, using a hypersonic aircraft dynamic model, also failed to converge to a solution. This experience, as well as the lack of suitable illustrative examples in the literature, motivated the construction of a less complex problem for which optimal periodic solutions could be obtained more easily. The assumption made here is that the characteristics of these optimal periodic processes are representative of those for the general class of optimal periodic control problems. The objective ultimately is to develop techniques suitable for solving problems such as the optimal cruise trajectory of an aircraft.

## 5.2 Sample Problem Description

A particular problem has been constructed to represent the general class of optimal periodic control problems with unrestricted period as defined in section 1.2. The sample problem is described in this section and local necessary conditions are formulated in terms of the parameters of the problem. The steady-state solution of the problem is examined for optimality using the frequency test.

A statement of the sample problem follows. Find the period, T, the scalar control, u(·), and the state, x(·), that minimize the performance index,

$$J = \frac{1}{T}\int_0^T \left(\frac{x_1^2}{2} + \frac{x_2^4}{4} - \frac{x_2^2}{2} + \frac{bu^2}{2}\right) dt \tag{5.1}$$

subject to the second order dynamic constraints,

$$\dot{x}_1 = x_2, \tag{5.2}$$

$$\dot{x}_2 = u, \tag{5.3}$$

and to the periodicity condition applied at the boundary,

$$x_1(T) = x_1(0), \tag{5.4}$$

$$x_2(T) = x_2(0). \tag{5.5}$$

The performance index (5.1) was constructed so that for particular conditions, periodic solutions would exist that provided better performance than the steady-state solution. The negative quadratic term provides the non-convexity in the performance criterion required to permit solutions other than the static solution. The quartic term dominates the quadratic term for large excursions of $x_2$, thereby bounding the minimizing solutions. The scalar parameter, b, weighting the control, determines the nature of the local optimal solutions as will be shown when the frequency test is applied.

First the variational Hamiltonian for this problem, defined in general by equation (2.2), may be written as

$$H = \frac{x_1^2}{2} + \frac{x_2^4}{4} - \frac{x_2^2}{2} + \frac{bu^2}{2} + \lambda_1 x_2 + \lambda_2 u, \tag{5.6}$$

where $\lambda_1$ and $\lambda_2$ are the Lagrange multipliers associated with the dynamic constraints (5.2) and (5.3). Reducing the Euler-Lagrange equations, (2.9) through (2.11), by solving for and eliminating the

control, u, results in the following set of nonlinear differential equations,

$$\dot{x}_1 = x_2, \tag{5.7}$$

$$\dot{x}_2 = -\frac{\lambda_2}{b}, \tag{5.8}$$

$$\dot{\lambda}_1 = -x_1, \tag{5.9}$$

$$\dot{\lambda}_2 = x_2 - x_2^3 - \lambda_1. \tag{5.10}$$

The periodicity conditions, (5.4) and (5.5), and the transversality condition (2.13), provide the boundary conditions, rewritten below, for this two point boundary value problem,

$$x_1(T) = x_1(0), \tag{5.11}$$

$$x_2(T) = x_2(0), \tag{5.12}$$

$$\lambda_1(T) = \lambda_1(0), \tag{5.13}$$

$$\lambda_2(T) = \lambda_2(0). \tag{5.14}$$

The form of the optimal periodic control is given by

$$u^o = \frac{-\lambda_2^o}{b}, \tag{5.15}$$

where $\lambda_2^o$ satisfies the above two-point boundary value problem. All solutions to this problem are extrema of the sample optimal periodic control problem. Also locally optimizing solutions of the problem must satisfy the additional transversality condition (2.14),

$$H = J^o. \tag{5.16}$$

This condition extracts from the previously obtained set of all extrema a subset of which each element has been extremized with respect to an unconstrained period.

Now consider the second order necessary conditions. By restricting the control weighting parameter such that $b > 0$, the Legendre-Clebsch condition is always satisfied in its strong form, i.e., $H_{uu} = b > 0$. The Weierstrass condition is also always satisfied since the Hamiltonian, (5.6), is regular, i.e., it has only one minimum with respect to u.

For the sample problem, there is only one steady state solution which satisfies all of the above necessary conditions for an optimal solution,

$$x_1 = x_2 = u = 0. \tag{5.17}$$

By using the frequency test [11], the range of the parameter, b, for which this static path is actually minimizing can be determined. The matrix expressions, corresponding to those in (2.22) of the frequency test, in terms of the sample problem are

$$f_x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad H_{xx} = \begin{bmatrix} 1 & 0 \\ 0 & 3x_2^2-1 \end{bmatrix},$$

$$\text{and} \quad H_{xu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{5.18}$$

where f is the functional representation of the dynamic constraints, (5.2) and (5.3). The Hermitian matrix (2.23), a form of the second

variation transformed into the frequency domain, becomes

$$\Pi(\omega) \;=\; f_u^{oT}\left[-j\omega I - f_x^o\right]^{-T} H_{xx}^o \left[j\omega I - f_x^o\right]^{-1} f_u^o + H_{uu}^o, \qquad (5.19)$$

after eliminating terms multiplied by the zero vector, $H_{xu}$. The matrix expression may be further reduced by introducing the remaining relationships from (5.18) into (5.19), which gives

$$\Pi(\omega) \;=\; \frac{1}{\omega^4} - \frac{1}{\omega^2} + b. \qquad (5.20)$$

As indicated in section 2.4, the steady state solution is minimizing if the matrix $\Pi(\omega)$ is non-negative definite for all values of the frequency, $\omega$. The minimum value for $\Pi(\omega)$ in (5.20) occurs when $\omega = \sqrt{2}$. As a result, when $b \geq \frac{1}{4}$, the matrix $\Pi(\omega)$ is non-negative definite, and the static solution is minimizing for this range of the parameter, b. However, in the range $0 \leq b < \frac{1}{4}$, there are frequencies for which the matrix is negative. For these values of b, there are periodic solutions which improve performance. The extraction and characterization of these solutions are important objectives of this work. Therefore, a thorough, numerical and analytical study of the sample problem, formulated in this section, is conducted in the remaining chapters of this work. But first, some special relationships resulting from the symmetry and order of the sample problem are developed in the last two sections of this chapter.

## 5.3 <u>The Trace of the Monodromy Matrix</u>

The trace of a matrix is an easily obtained and a quite

powerful indicator of some intrinsic properties of a matrix. Because of the relationship of the trace and the eigenvalues of a matrix, the stability of a system associated with the eigenvalues of the matrix characterizing the system is frequently determined by obtaining its trace. As was derived in section 4.2 for the general optimal periodic control problem, locally optimizing solutions, except for isolated cases, belong to the set of unstable, periodic solutions to the Euler-Lagrange equations, (2.9) through (2.11). For the case of a fourth order, symmetric system, this result is demonstrated by a very simple relationship involving the trace of its monodromy matrix and elements of the Riccati variable associated with the system.

The sample problem formulated in this chapter generates a fourth order system of differential equations, (5.7) through (5.10). The system is symmetric, satisfying the conditions for symmetry given in appendix B by equation (B.2). The monodromy matrix for the sample problem, as well as for any other fourth order, symmetric system, may be expressed by,

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} \doteq \begin{bmatrix} a & b & e & f \\ c & d & -f & g \\ h & i & a & -c \\ -i & j & -b & d \end{bmatrix}. \tag{5.21}$$

This special form of the monodromy matrix for a symmetric system is derived in appendix B.

It is apparent from examining the monodromy matrix (5.21) that only its first two columns need be determined to calculate the

trace of the matrix,

$$\text{trace}\ \Gamma = 2(a + d). \tag{5.22}$$

Using the relationship, (3.48), between the trace and the eigenvalues of a matrix, the reciprocity property of the eigenvalues of the transition matrix (section 3.3), and the property that at least two eigenvalues of the monodromy matrix are unity (section 3.4) permits expressing the trace of the monodromy matrix (5.22) in terms of one unknown eigenvalue, $\mu$,

$$2(a + d) = 1 + 1 + \mu + \frac{1}{\mu}. \tag{5.23}$$

From the similarity transformation in Theorem 4.1, another expression for the trace of the monodromy matrix may be written as follows,

$$\text{trace}\ \Gamma = \text{trace}\left[\Gamma_{11} + \Gamma_{12}P\right] + \text{trace}\left[\Gamma_{22} - P\Gamma_{12}\right], \tag{5.24}$$

where P is the initial value matrix of the Riccati variable. It was shown in section 4.2 that this similarity transformation also partitions the eigenvalues of $\Gamma$ into the two submatrices on the diagonal of the transformed matrix. The eigenvalues of the submatrix, $[\Gamma_{11} + \Gamma_{12}P]$, are the reciprocal of the eigenvalues of the other submatrix, $[\Gamma_{22} - P\Gamma_{12}]$. As a result, an expression for the trace of the submatrix $[\Gamma_{11} + \Gamma_{12}P]$ may be written as

$$\text{trace}\left[\Gamma_{11} + \Gamma_{12}P\right] = 1 + \mu. \tag{5.25}$$

Since the Riccati matrix is symmetric, it can be expressed in terms of its scalar elements as follows,

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}. \tag{5.26}$$

Using equations (5.24) and (5.26) to perform the matrix operations indicated by equation (5.25) gives the result,

$$a + d + eP_1 + gP_3 = 1 + \mu. \tag{5.27}$$

Subtracting equation (5.27) from (5.23) gives

$$a + d - eP_1 - gP_3 = 1 + \frac{1}{\mu}. \tag{5.28}$$

Eliminating the eigenvalue $\mu$ from the two equations (5.27) and (5.28) and then rearranging the resulting expression provides the following useful relationship,

$$(a + d - 1)^2 - (eP_1 + gP_3)^2 = 1. \tag{5.29}$$

This equation clearly isolates the range of values that the trace of the monodromy matrix may assume when the elements of the Riccati matrix are real. This leads to a necessary condition for local optimality of extremal solutions to the optimal periodic control problem which may be expressed in terms of elements of the trace of the monodromy matrix,

$$a + d \geqq 2. \tag{5.30}$$

The necessary condition (5.30) for fourth order, symmetric systems

excludes any solution whose eigenvalues lie on the unit circle at points other than the critical points, (1,1) or (-1,-1). This is in full agreement with the more general results derived in section 4.2.

## 5.4 A Solution to the Riccati Equation

Another important condition for optimality involves the existance of the Riccati variable over the full time duration spanned by the optimal solution. When the solution is periodic, only an interval of one period need be considered. In this section the algebraic Riccati equation, derived in section 4.1, is evaluated for fourth order, symmetric systems. Two initial value matrices for the Riccati variable are derived in terms of elements of the monodromy matrix. These results explicitly determine the only periodic solutions to the Riccati matrix differential equation (2.15). With the initial conditions, the existance of both sets of Riccati variables over a full period may be examined by integrating the differential equation.

The algebraic Riccati equation (4.16) provides the general relationship between the elements of the monodromy matrix and the initial conditions for periodic solutions to the Riccati differential equation. This relationship may be specialized for fourth order, symmetric systems by expanding the algebraic equation, using the particular expressions for the monodromy matrix and the Riccati matrix given by equations (5.21) and (5.26). The results of this matrix equation expansion are expressed below in the equivalent form of four simultaneous, algebraic scalar equations,

$$eP_1^2 + gP_2^2 + 2cP_2 - h = 0, \tag{5.31}$$

$$-fP_2^2 + eP_1P_2 + fP_1P_3 + gP_2P_3 + (d - a)P_2 + bP_1 + cP_3 - i = 0, \tag{5.32}$$

$$fP_2^2 + eP_1P_2 - fP_1P_3 + gP_2P_3 + (a - d)P_2 + bP_1 + cP_3 + i = 0, \tag{5.33}$$

$$eP_2^2 + gP_3^2 + 2bP_2 - j = 0. \tag{5.34}$$

Because the Riccati matrix is symmetric, there are three unknown elements in the matrix. Therefore, only three of the above equations are independent.

A particular combination of these equations results in an explicit solution for the initial condition of the Riccati variable, $P_2$, expressed in terms of elements of the monodromy matrix only. Subtracting equation (5.33) from equation (5.32) and multiplying the result by the monodromy matrix elements e and g produces the following equation,

$$f(2egP_1P_3) - 2efgP_2^2 + 2eg(d - a)P_2 - 2egi = 0. \tag{5.35}$$

Multiplying equation (5.31) by the elements e and f and adding the result to that of multiplying equation (5.34) by the elements f and g gives a second equation,

$$f(e^2P_1^2 + g^2P_3^2) + 2efgP_2^2 + 2f(ce + bg)P_2 - f(eh + gj) = 0. \tag{5.36}$$

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The unknowns, $P_1$ and $P_3$, can be eliminated by adding the two equations, (5.35) and (5.36), and replacing the term involving $P_1$ and $P_3$ in the result with its equivalent expression,

$$(eP_1 + gP_3)^2 = (a + d - 1)^2 - 1, \tag{5.37}$$

derived in the previous section. This results in a simple, algebraic expression for $P_2$ involving only the elements of the monodromy matrix,

$$P_2 = \frac{2egi + f[eh + gj + 1 - (a+d-1)^2]}{2\,[f(ce + bg) + eg(d - a)]} \;. \tag{5.38}$$

It is interesting that the initial condition for this component of the Riccati matrix is single-valued. This implies that there is only one starting condition, $P_2$, associated with each initial condition, $t_0$, for the optimal periodic control problem, that results in a periodic solution to the corresponding Riccati differential equation.

Expressions for the initial conditions $P_1$ and $P_3$ can now be obtained by using the value for $P_2$ from equation (5.38) in equations (5.31) and (5.34), rearranged below,

$$P_1 = \pm\left[\frac{h - 2cP_2 - gP_2^2}{e}\right]^{\frac{1}{2}} , \quad \text{and} \tag{5.39}$$

$$P_3 = \pm\left[\frac{j - 2bP_2 - eP_2^2}{g}\right]^{\frac{1}{2}} . \tag{5.40}$$

The indeterminacy of the corresponding signs in equations (5.39) and (5.40) may be alleviated by the following expression relating $P_1$ and $P_3$,

$$P_3 = -\frac{eP_2 + b}{gP_2 + c} P_1. \tag{5.41}$$

This relationship was obtained by adding equations (5.32) and (5.33) and rearranging the result.

The expressions (5.38) through (5.41) provide all of the initial conditions which result in a periodic solution to the Riccati differential equation, (2.15). In general, and at most, there are two sets of these starting conditions. Corresponding to each set is a single set of periodic solutions which satisfy the differential equation. It may be deduced from Theorem 4.1 that the two periodic matrix solutions have properties that are related to the pairing of the eigenvalues in the submatrices on the diagonal of the transformed matrix (4.19). These results are explicitly shown in the numerical investigation of the next chapter.

# CHAPTER 6

# NUMERICAL INVESTIGATION

A numerical investigation of the sample optimal periodic control problem illustrates the extraordinary complexity of this class of optimization problem. Periodic solutions to the Euler-Lagrange equations are obtained in the first section that are continuously related forming a one-parameter family of solutions. In section two, particular solutions are identified as bifurcation points by a measure of their stability. Additional families of solutions are obtained that branch from the original family at the bifurcation points, and characteristics that distinguish their solutions are examined. Local minimal solutions are determined in the next section. The performance index associated with each periodic extremum is computed and the necessary condition for optimization with respect to the period is applied. In section four, the sufficiency condition associated with the existence of the Riccati variable is examined. Finally, in the last section the periodic regulator which was developed in chapter four is demonstrated using the local optimal periodic solution associated with the principal family.

## 6.1 Principal Family of Periodic Solutions

The Euler-Lagrange equations in the form of a two-point boundary value problem, equations (5.7) through (5.14), are solved by

numerical techniques in this section. This problem was constructed to be as simple as possible and still produce periodic solutions. However, the resulting solution to the problem is remarkably complex. An infinity of periodic solutions is shown to exist for the problem forming one-parameter families. A distinguishing characteristic of solutions in a family is the number of positive crossings of the axis in one period. The principal family, which originates at the static equilibrium solution of the problem, is determined and some characteristics of its solutions are examined.

An existing computer program, utilizing a "shooting method", was used to compute the periodic solutions obtained in this work. The program originally was developed by Roger A. Broucke in his study of periodic orbits near equilibrium points in the restricted problem of three bodies [24]. It was designed to find periodic orbits (solutions) for any fourth order Hamiltonian system, given a set of initial conditions within the range of convergence for the system. Suitable initial conditions may be found using one of several search routines available in the program. After determining a periodic solution, the program uses an interpolation routine to predict initial conditions for a neighboring periodic solution. In this manner it determines and follows a family of periodic solutions. The scheme is analogous to the principal of analytic continuation, described by Szebehely [23], and it exploits the inherent properties of the Hamiltonian system and its associated monodromy matrix.

The periodic solutions obtained for this problem are uniquely identified by their initial conditions as was expressed previously

in chapter three. This provides a very useful means of categorizing results and, in the case of fourth order systems, illustrating them. In figures 6.1a and 6.1b, plots of the principal family of periodic solutions are shown. Each point of the plots represents the initial conditions of a periodic solution to the Euler-Lagrange equations of the sample problem. In this case, the initial conditions for the variables $x_2$ and $\lambda_1$, are arbitrarily chosen to be zero. This results from the order of the system of differential equations being reducible by two as expressed in chapter three. The remaining two initial conditions are $x_1$, the abscissa of both graphs, and $\lambda_2$, implied by the Hamiltonian in figure 6.1a and explicitly identified on the ordinate in figure 6.1b. A control weighting parameter of $b = 0.1$ has been used in all examples of this chapter.

The period of the solutions comprising the principal family ranges continuously from approximately 2.11 to 5.92. The two extrema correspond to the fast and slow frequencies represented by the eigenvalues of the Euler-Lagrange equations linearized about the static equilibrium solution. This relationship to the period is shown for $x_1$ in figure 6.2a and for H in figure 6.2b. Traversing the right half plot of figure 6.1a in a counter-clockwise direction, beginning at the origin, corresponds to solution with increasing period in figures 6.2a and 6.2b.

## 6.2 Bifurcation Points and New Families

The stability of a periodic solution is an important inherent characteristic of the solution. It is also a key element in the

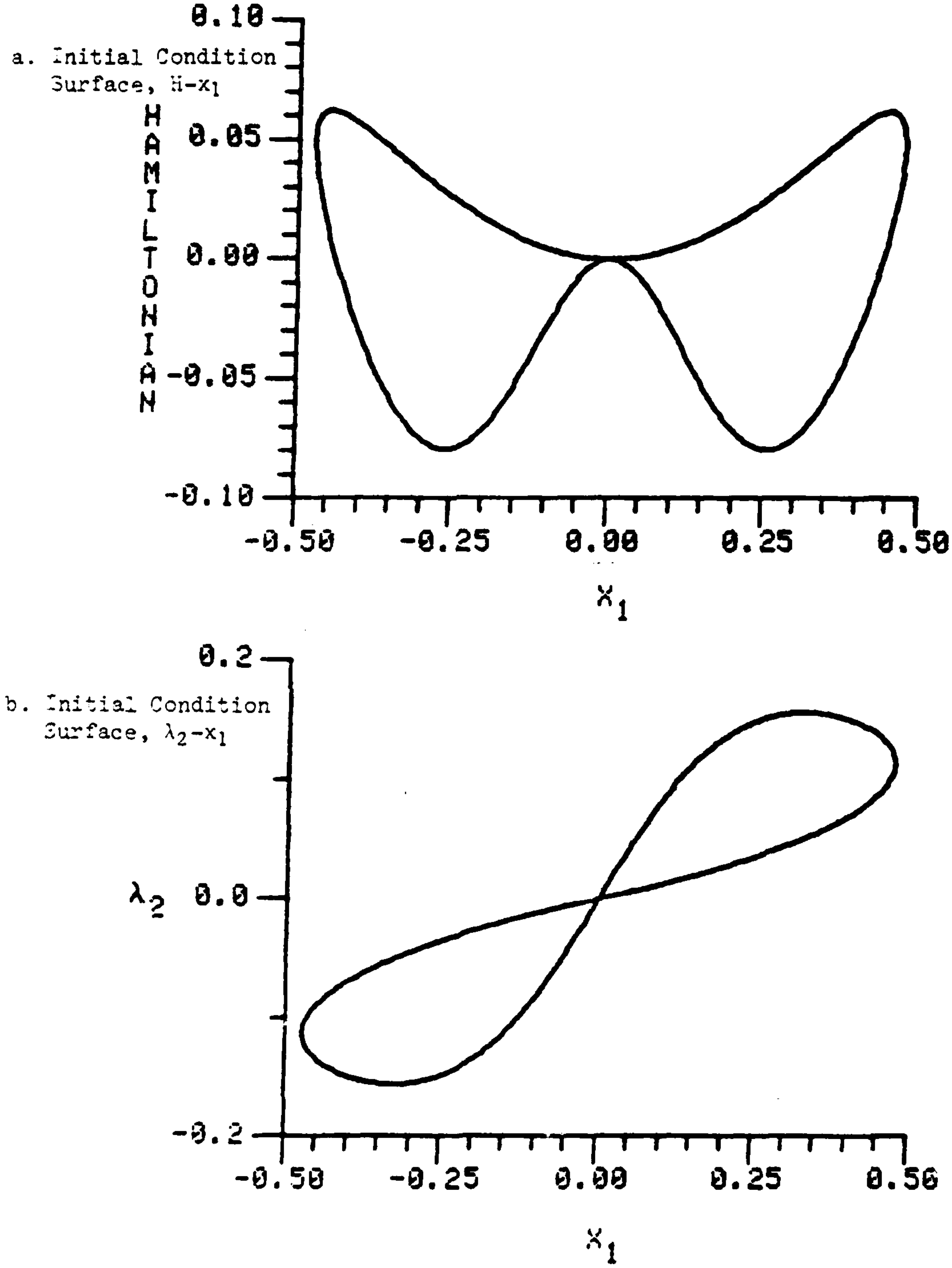



Figure 6.1 PRINCIPAL FAMILY SOLUTIONS

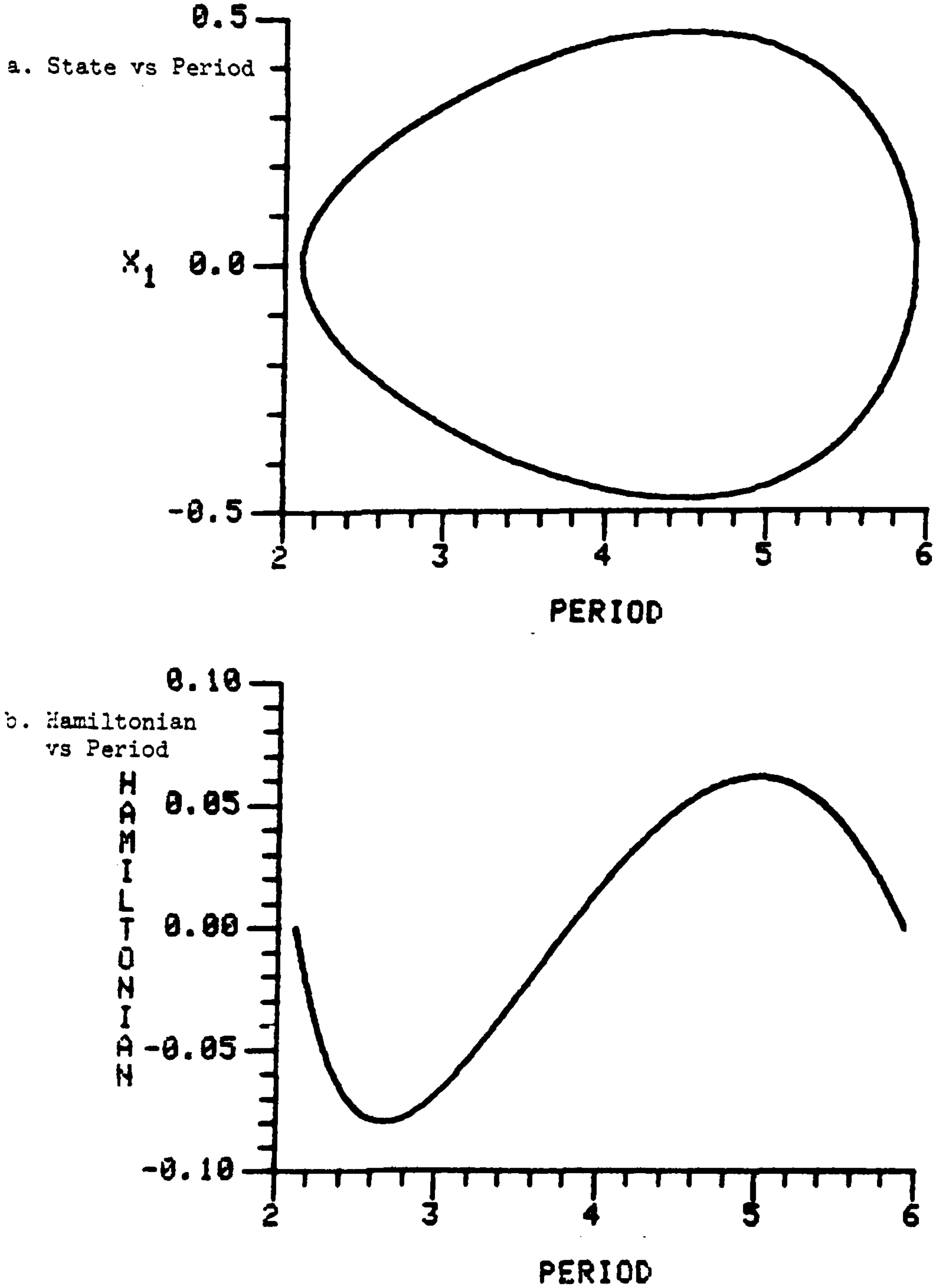



Figure 6.2 PRINCIPAL FAMILY SOLUTIONS

relationship between families of solutions. In this chapter, as in previous chapters, a periodic solution is stable if no eigenvalue of its monodromy matrix lies outside the unit circle. Otherwise, it is unstable. A measure of the stability of a periodic solution is the stability index, k, which is defined as the trace of the monodromy matrix of the solution less two, for a fourth order system. An equivalent definition is k equals the sum of the remaining eigenvalues of the monodromy matrix after eliminating the two unity eigenvalues.

The relationship of the stability index to the stability of a periodic solution for a fourth order system is identified by the following inequalities:

$$k \leqq 2, \quad \text{stable solution;} \tag{6.1}$$

$$k > 2, \quad \text{unstable solution.} \tag{6.2}$$

It is interesting to note that the stability index, k, is precisely that quantity, a + d, which occurs in the inequality equation (5.30) expressing a necessary condition for optimality of a symmetric, fourth order periodic control system.

An important relationship used by Hénon [27] and Contopoulos [26] expresses properties of the solutions of a family in terms of a critically stable solution, i.e., all eigenvalues equal one. The relationship may be written in terms of critical values of the stability index,

$$k_c = 2\cos(2\tfrac{m}{n}\pi), \tag{6.3}$$

where m and n are both integers associated with properties of periodic solutions with stability index equal to $k_c$. One importance of this relationship is that solutions of a family which may be common to other branching families can be identified by their stability index. These common solutions are frequently called bifurcation points. Some points for the principal family are plotted in figure 6.3 and are identified by the notation n/m Fi, where n and m are the integers above (/m is omitted when m is one), F is the family designator, and i distinguishes equal critical values. The stable and unstable solutions of the family are identified by a broken line and a solid line respectively. The unstable solutions have no bifurcation points, according to equation (6.3), and they represent extrema of the problem which also satisfy the necessary condition for optimality, equation (5.30).

Branching families are found by searching for new periodic solutions in the vicinity of bifurcation points. The new families are then traced in the same manner as the principal family. A few new families branching from the principal family are shown in figure 6.4. Stable and unstable solutions are again identified by broken and solid lines respectively. Only the lower half of the plots are shown because this region provides the solutions of most interest to the optimal periodic control problem. Solutions that satisfy the necessary condition for optimal period, $H = J^o$, and improve performance compared to the steady state must lie in the region $H < 0$. New families are identified using their generating bifurcation point identifier preceded by an F.

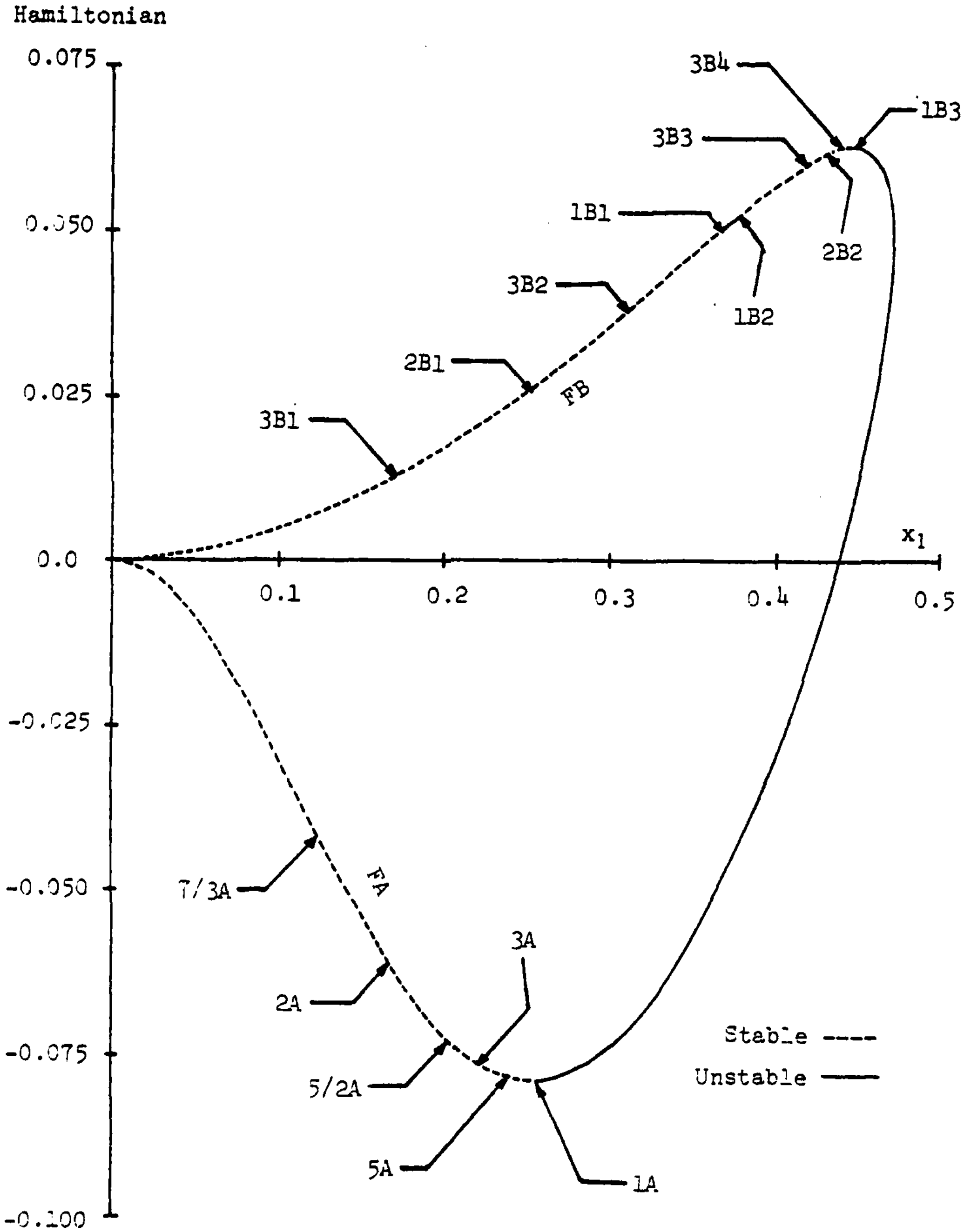



Figure 6.3 STABILITY REGIONS AND BIFURCATION POINTS PRINCIPAL FAMILY

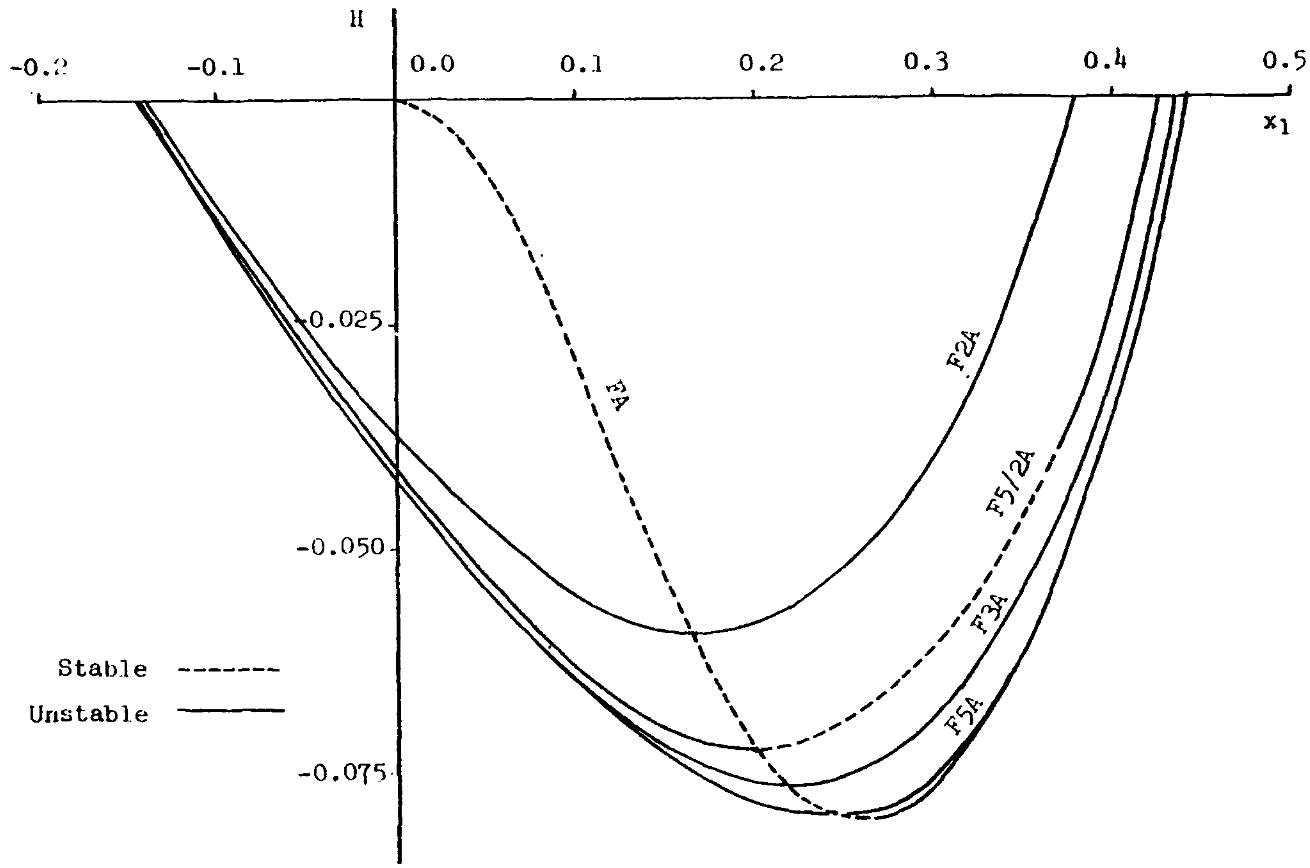
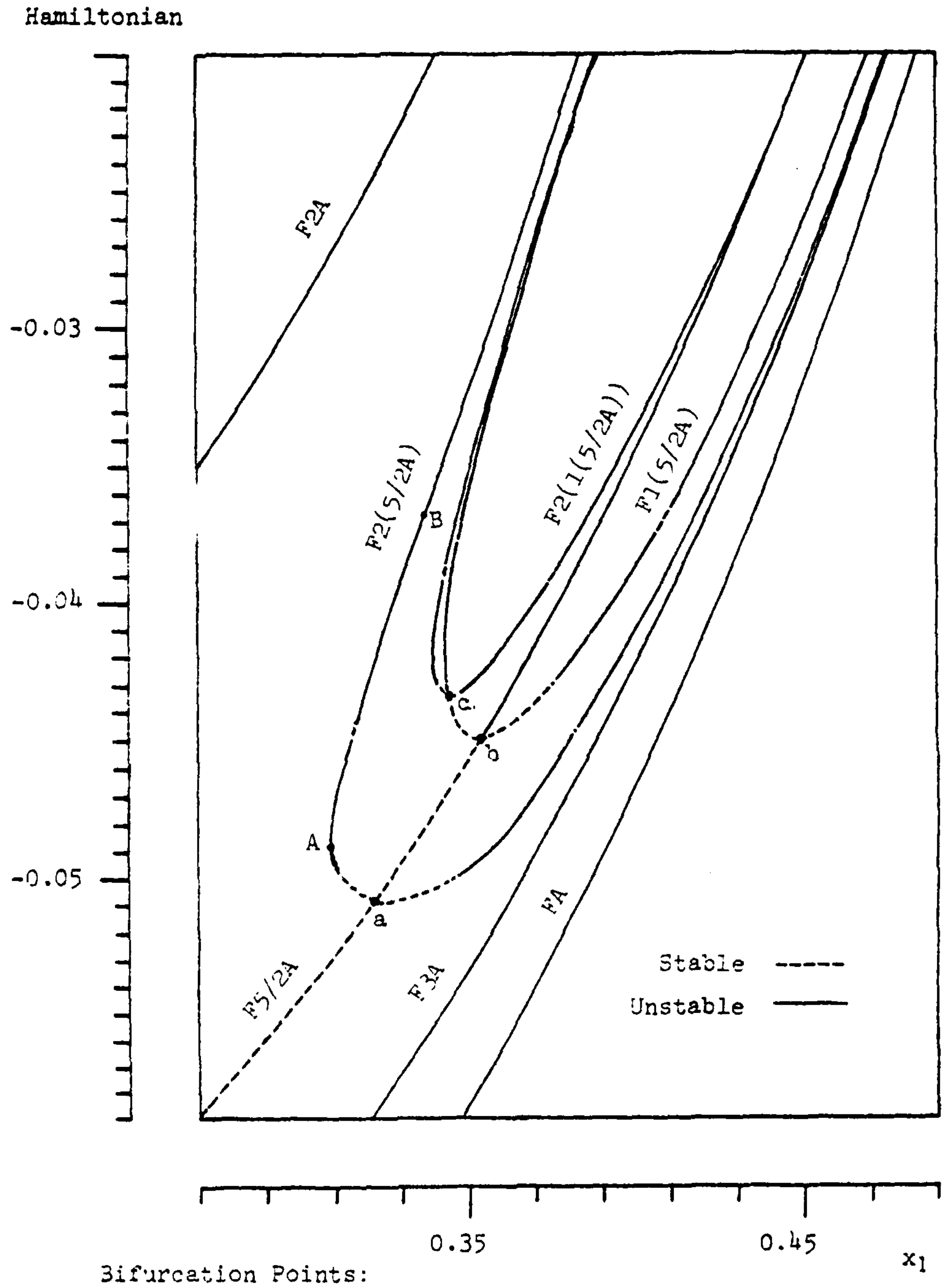



Figure 6.4 PRINCIPAL FAMILY AND BRANCH FAMILIES

Two additional levels of branching are shown on the plot in figure 6.5, which is a detailed enlargement of a section of the previous plot. This process may be continued indefinitely, limited only by the finite word length of the computer and the persistence of the investigator. It is quite apparent that large regions of the phase space are densely packed with families of periodic solutions, each an extremum of the optimal periodic control problem.

Before proceeding to the next section and further investigating the sample problem, some important characteristics of the periodic solutions of a family are examined. A characteristic that distinguishes one family from another is the number of axis crossings that occur in the same direction during one period of a solution. This is frequently expressed as the number of arcs of a solution. Referring back to the equation (6.3), the integer, n, is the ratio of the arcs of a solution on the branching family to the arcs of a solution on the principal (or branched from) family.

This characteristic is easily discernible in the following sequence of time history plots and phase plots of solutions representing different families. In figure 6.6, a time history plot of the variable $x_1$ for three solutions on the principal family are shown. The plots a, b, and c correspond to solutions at the bifurcation points 2A, 5/2A, and 1A. This sequence illustrates the variation of $x_1$ in amplitude and period along the principal family FA. In figure 6.7, a similar sequence of plots is shown for the branching family F2A. The evolution of a two arc solution from two periods of a one




Bifurcation Points:
a. 2(5/2A) b. 1(5/2A) c. 2(1(5/2A))

Figure 6.5 BRANCH FAMILIES (Detail)

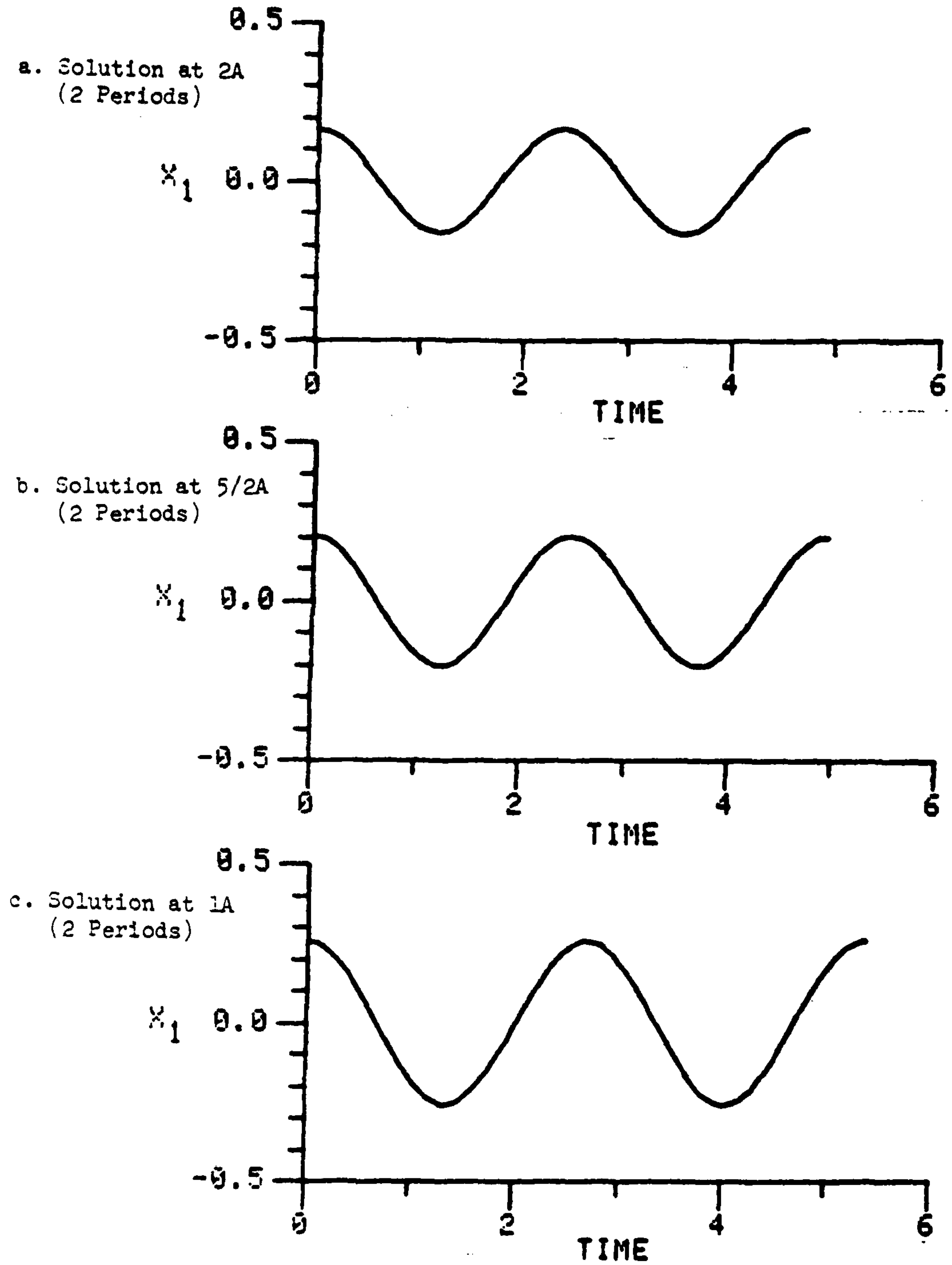



Figure 6.6 SOLUTIONS ON FAMILY - FA

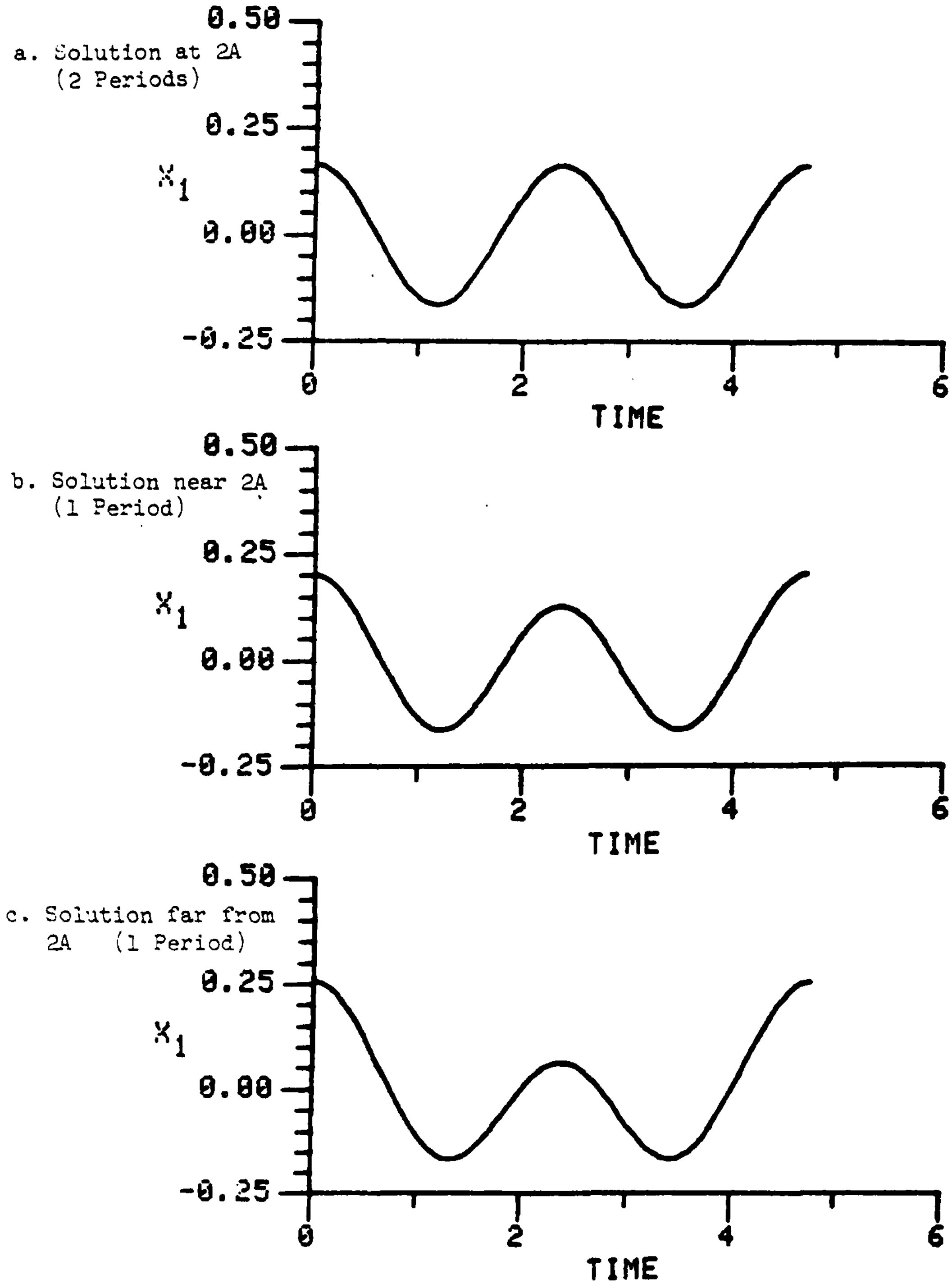



Figure 6.7 SOLUTIONS ON FAMILY - F2A

arc solution beginning at the bifurcation point 2A is identifiable in the sequence of plots a, b, and c. The same sequence of plots for the family F5/2A is depicted in figure 6.8. Plot a, corresponds to five periods of the one arc solution at the bifurcation point 5/2A. Plot c, corresponds to the solution at the bifurcation point 2(5/2A) as shown on the branching family identified in the detailed enlargement, figure 6.5. Plot b, is an intermediate solution on the family. The final time history sequence of plots, figure 6.9, shows the family F2(5/2A). Two periods of the five arc solution at the bifurcation point 2(5/2A) are shown in plot a. The remaining two plots, b and c, show the evolution of a ten arc solution.

A useful and probably more frequently used plot of periodic solutions is one providing phase space relationships. In figure 6.10, phase plots relating the variable $x_2$ to $x_1$, corresponding to the families FA, F2A, F5/2A, and F2(5/2A) are shown in plots a through d respectively. Each of the four plots represent the corresponding solutions in plot c, of the previous time history plots.

Considerable additional work has been completed by Hénon [27] and Contopoulos [26] classifying bifurcation points. For certain conditions, which exist in this problem, trifurcations are predicted. Frequently, one of the two or three new families is composed of non-symmetric solutions. For such a case the family does not lie in the symmetric plane $(x_1, 0, 0, \lambda_2)$. Further classification of critical solutions of the first kind ($k_c = +2$) and the second kind ($k_c = -2$) are summarized in appendix C.

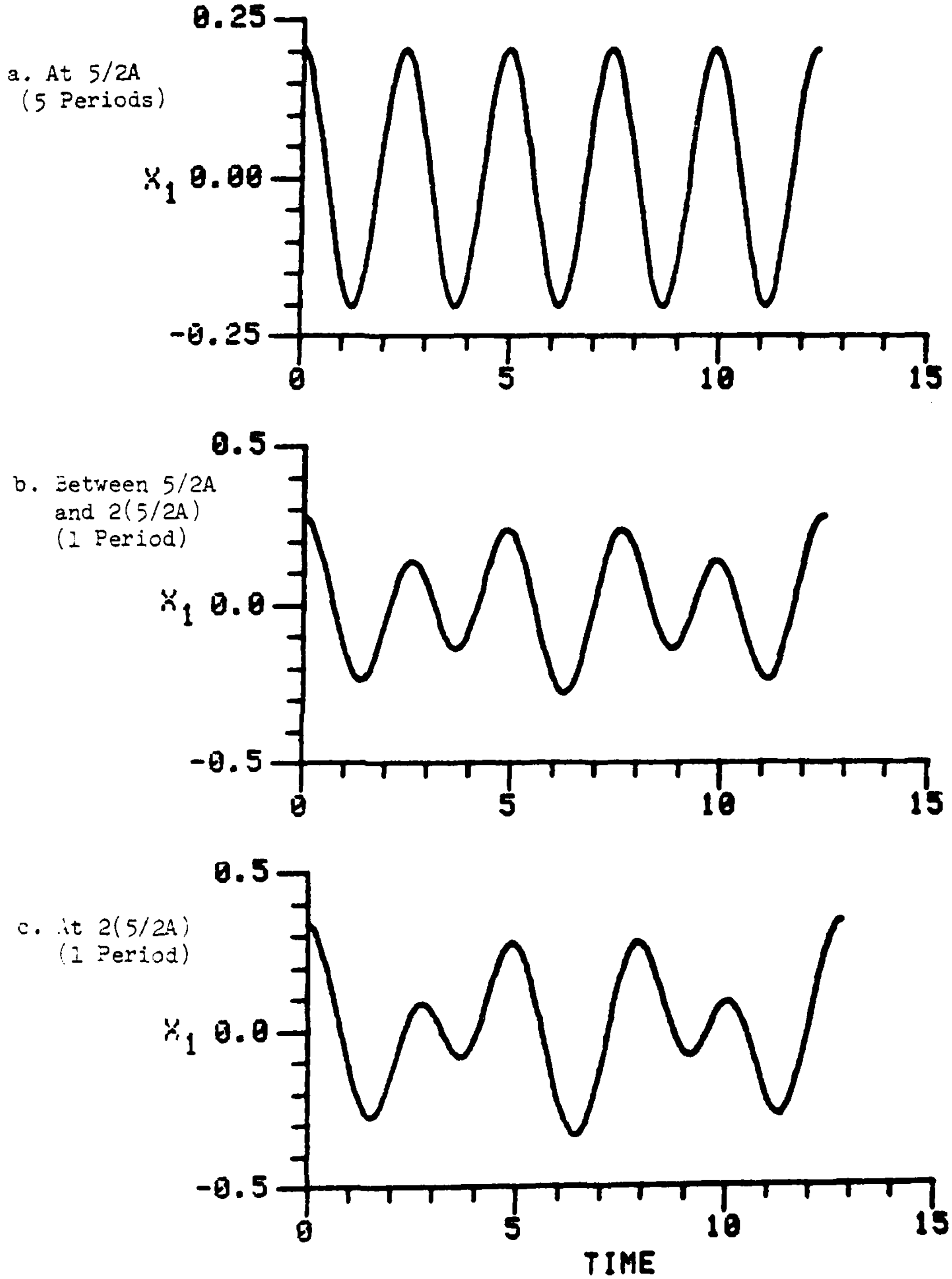
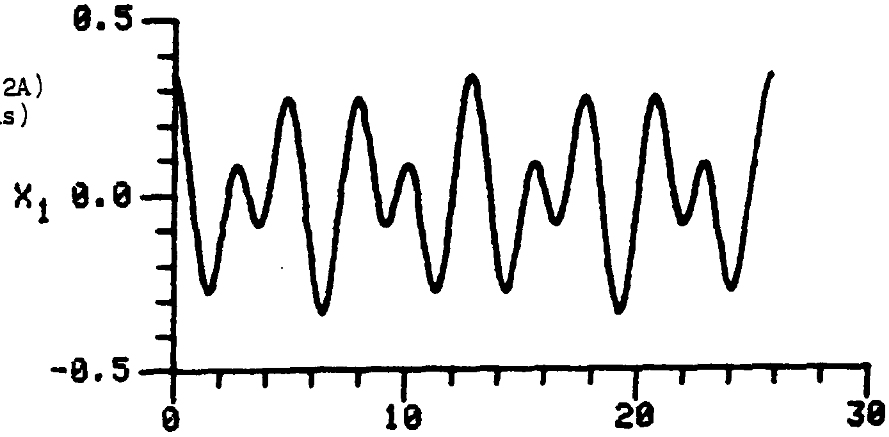
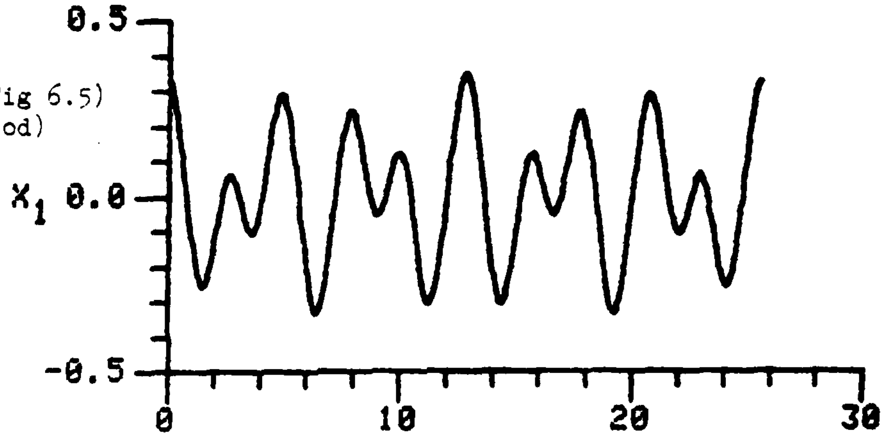



Figure 6.8 SOLUTIONS ON FAMILY - F5/2A

a. At 2(5/2A)
(2 Periods)




b. At A (Fig 6.5)
(1 Period)




c. At 3 (Fig 6.5)
(1 Period)

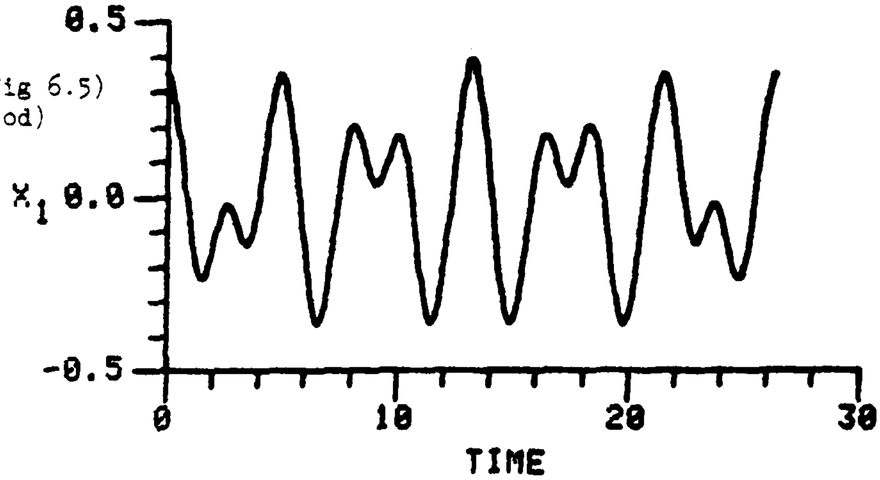



Figure 6.9 SOLUTIONS ON FAMILY - F2(5/2A)

a. Family FA

b. Family F2A

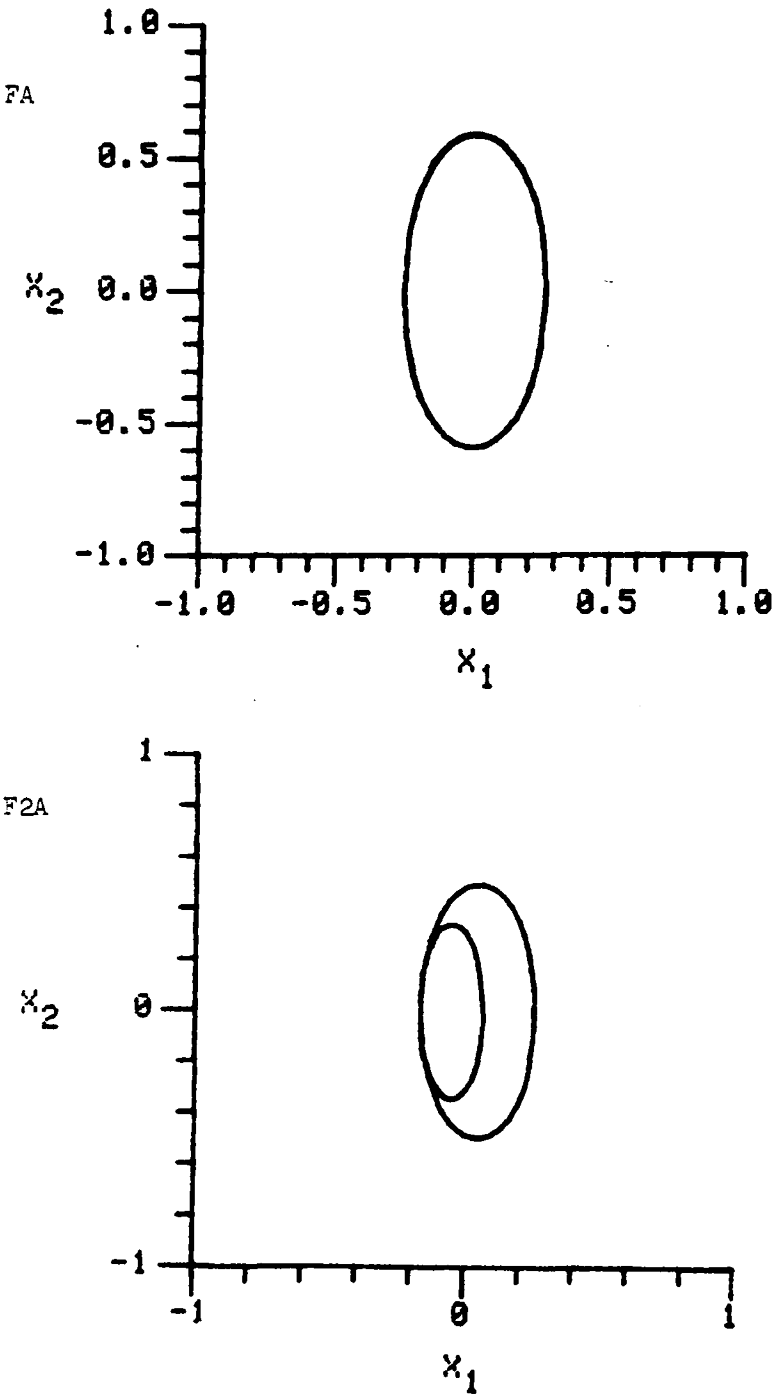
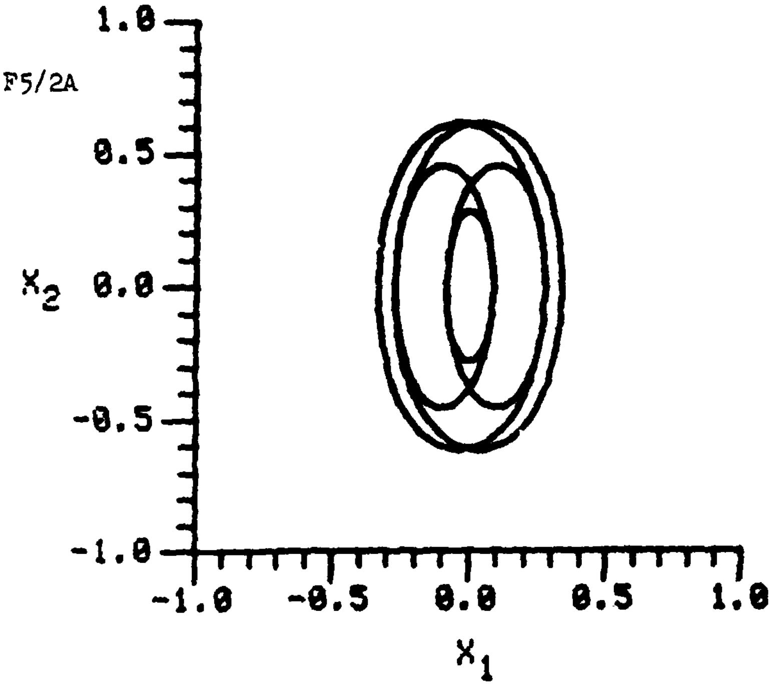



Figure 6.10 PHASE PLOTS

c. Family F5/2A




d. Family F2(5/2A)

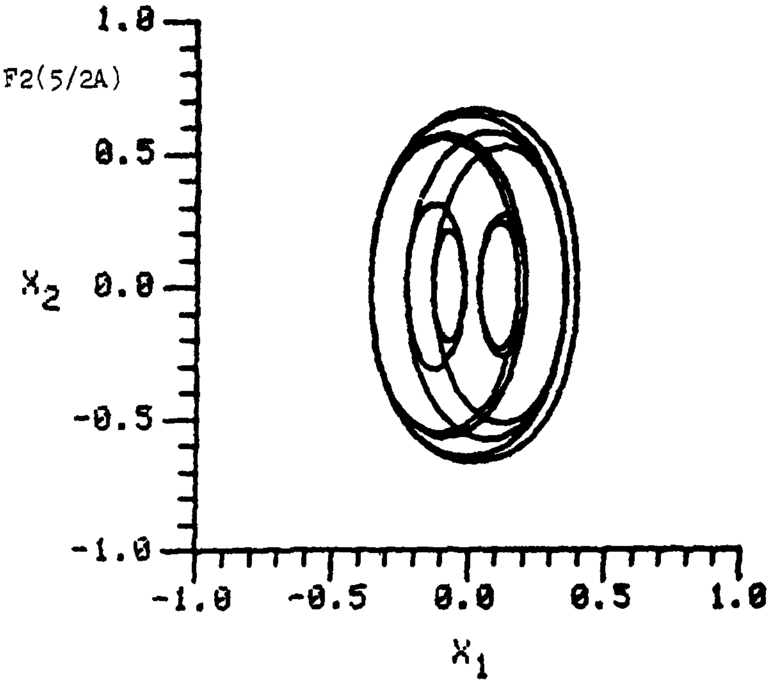



Figure 6.10 PHASE PLOTS (continued)

## 6.3 Optimized Solutions for Free Period

The necessary condition for an optimal periodic solution when its period is unconstrained is applied in this section to the extremal solutions obtained in the previous sections. This condition screens the infinity of solutions forming families so that only individual solutions of the families remain. Some interesting observations from the results obtained that pertain to the performance index are expressed concluding the section.

The performance index is calculated for solutions of the principal family and the results are plotted in figure 6.11. For these plots the solid line represents the initial conditions of periodic solutions to the Euler-Lagrange equations as in figure 6.1a, where the initial conditions, $x_2$ and $\lambda_1$, are zero, the initial condition $x_1$ is directly determined on the abscissa, and the initial condition $\lambda_2$ is implied by H on the ordinate. The broken line identifies on the ordinate the performance index, associated with each periodic solution corresponding to the initial conditions represented by $x_1$. At the intersection of the two plots is the solution for which the condition $H = J^o$ is satisfied. The upper half of the plots, $H > 0$, has been omitted to simplify the figure. Similarly, results for family F1A are plotted in figure 6.12. The expanded ordinate in figure 6.12b better illustrates the extrema. The two apparent local minima (excluding those resulting from the symmetry of the problem) for this family are actually the same solution intersecting the surface of initial conditions of the plot in two points.

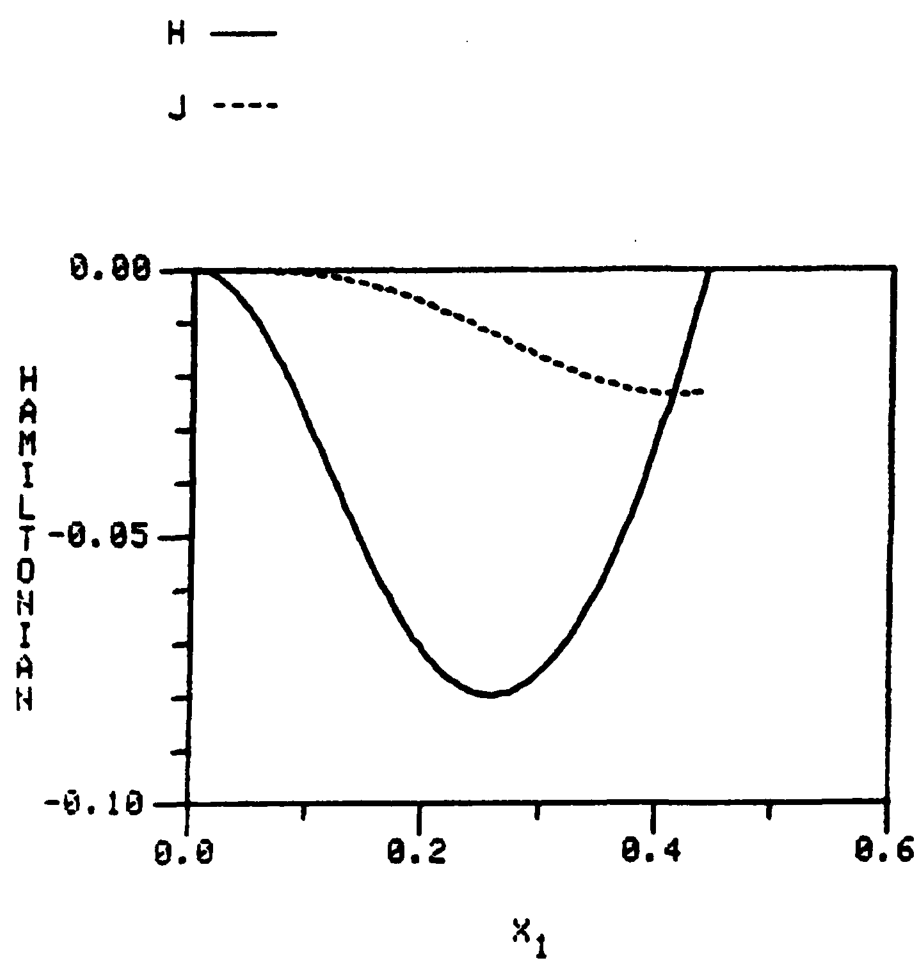



Figure 6.11 APPLICATION OF OPTIMAL PERIOD CONDITION
PRINCIPAL FAMILY

a. Full scale

b. Detail

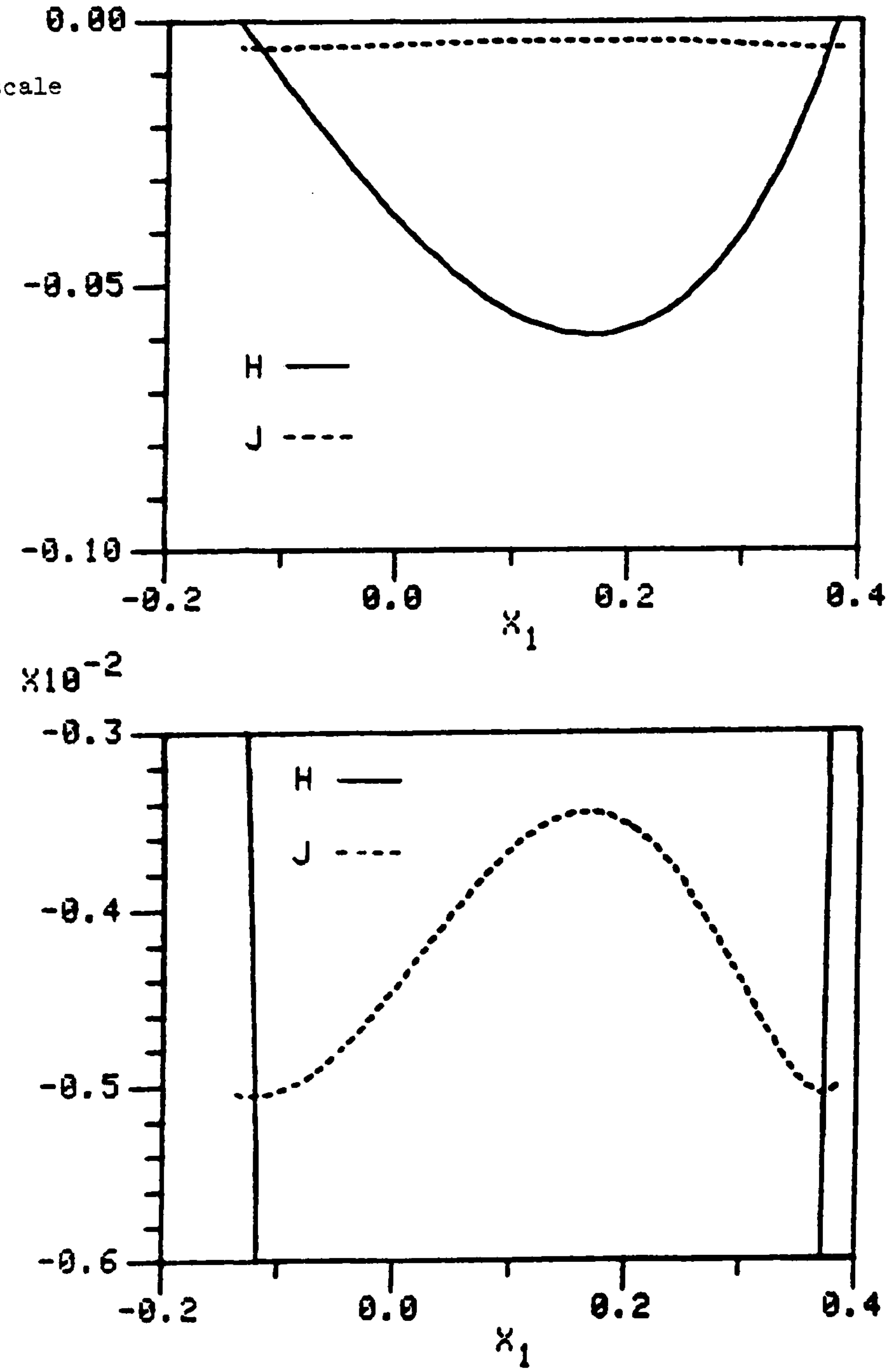



Figure 6.12 APPLICATION OF OPTIMAL PERIOD CONDITION
FAMILY F2A

The performance index as a function of the period is shown in figure 6.13 on plot a for the principal family FA and on plot b for the branching family F2A. The extrema of the performance index, determined from the plots in figure 6.13, agree with the corresponding results in figures 6.11 and 6.12.

Periodic solutions satisfying all the first order necessary conditions for a local optimum with unconstrained period are plotted for the principal family FA and the branch families F2A, F3A, and F5/2A. In figure 6.14 phase plots relating the variable $x_2$ to $x_1$ are shown for the four local minimum solutions. A time history of the control variable corresponding to each of the previous four local minimum solutions is plotted in figure 6.15. The initial conditions, optimal period, and performance index for the local minima of the four families are summarized in table 6.1.

Concluding this section are three interesting observations made during the numerical investigation which may or may not be true in general. First, every periodic solution to the Euler-Lagrange equations obtained in this investigation provided an improvement when compared to the performance index of the steady-state solution. Second, the performance index is a relative maximum for the branch family when the Hamiltonian is also an extremum with respect to the initial condition. Finally, there are no solutions with period less than that associated with the fast frequency eigenvalue (from linear analysis about the static equilibrium point) that improves the steady-state performance.

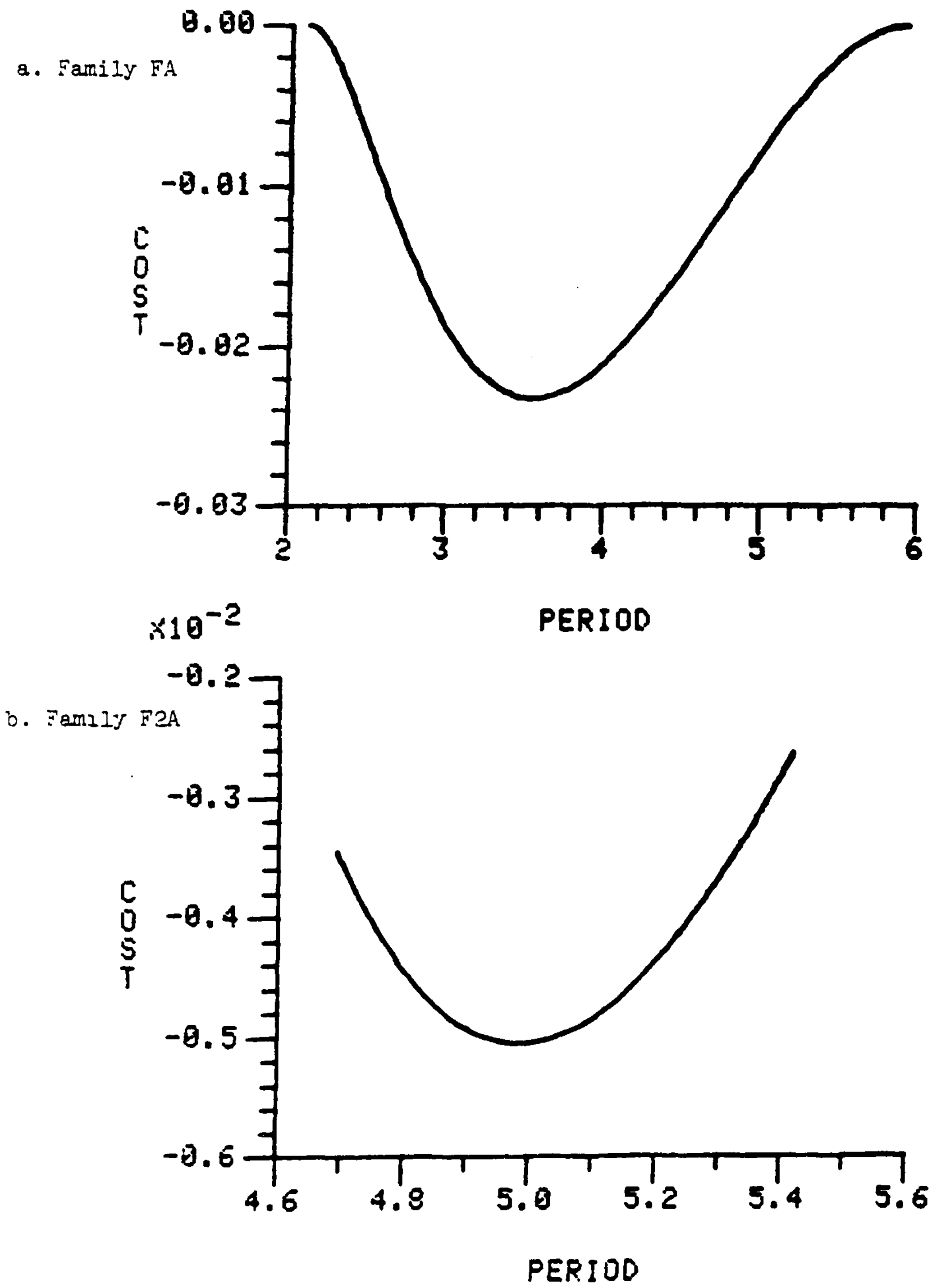



Figure 6.13 RELATIONSHIP OF PERFORMANCE TO PERIOD

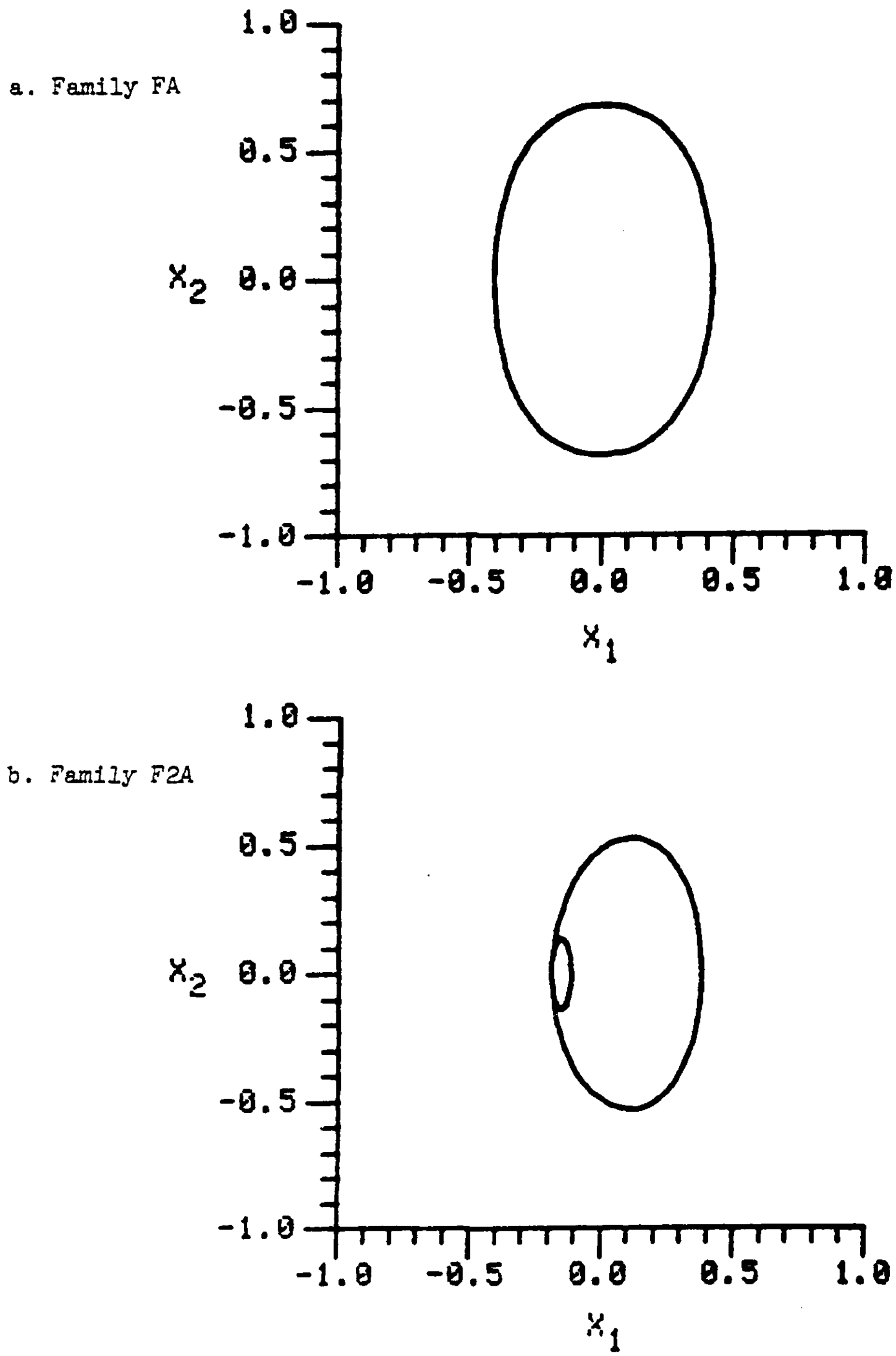
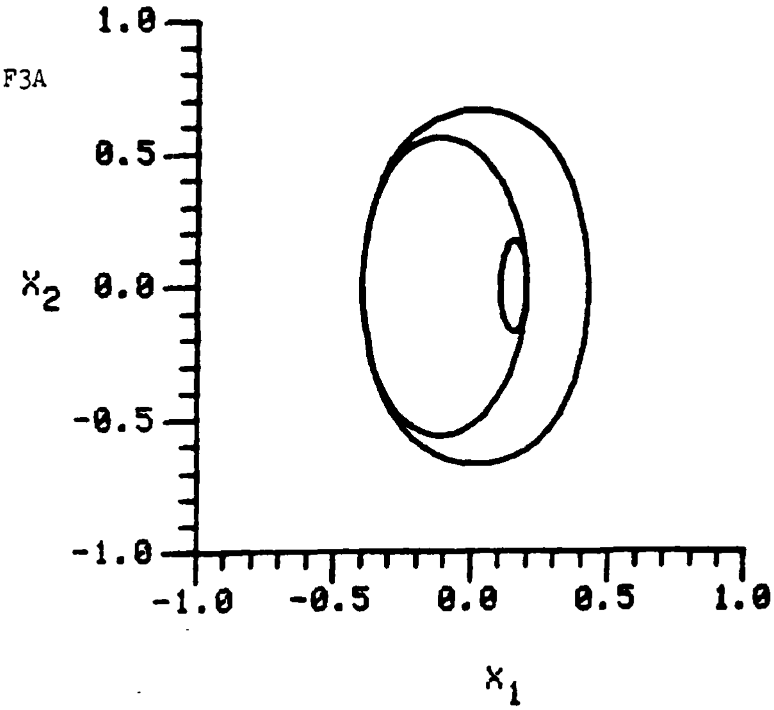



Figure 6.14 EXTREMA SATISFYING OPTIMAL PERIOD CONDITION

c. Family F3A




d. Family F5/2A

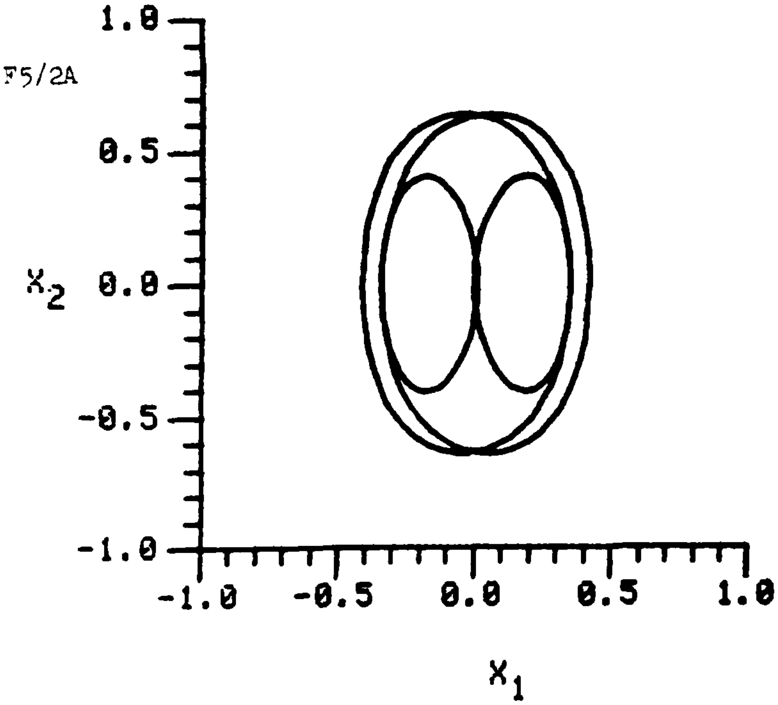
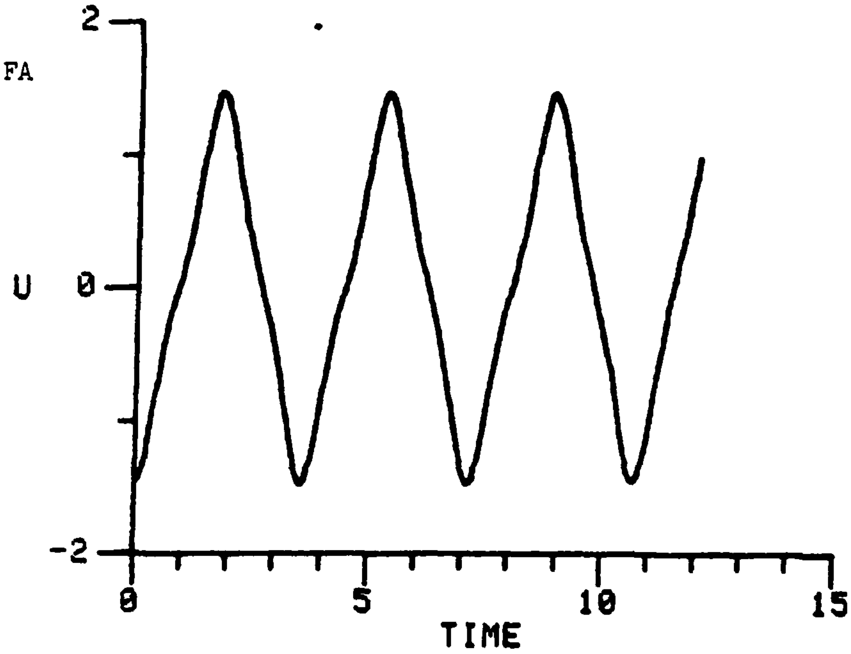



Figure 6.14 EXTREMA SATISFYING OPTIMAL PERIOD CONDITION (Continued)

a. Family FA




b. Family F2A

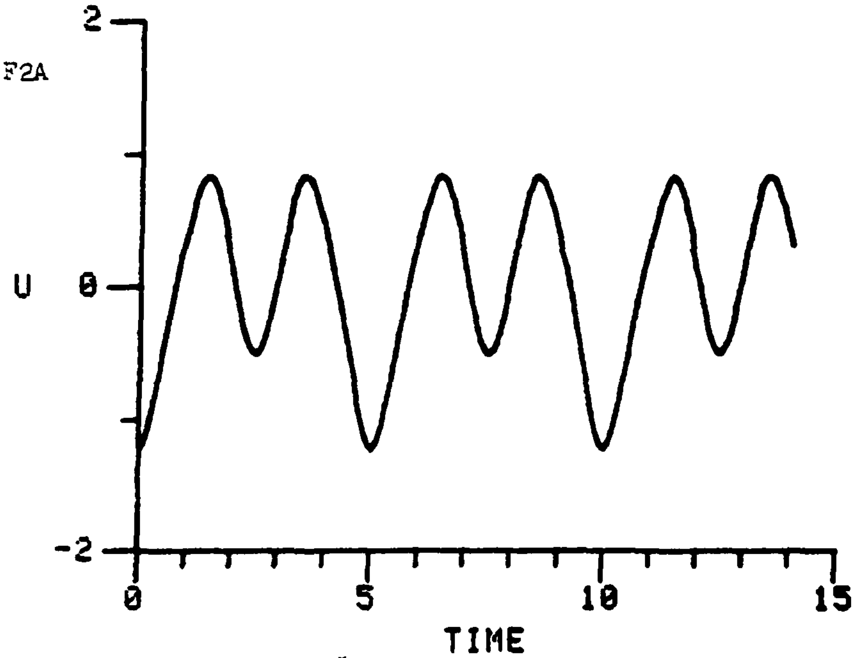
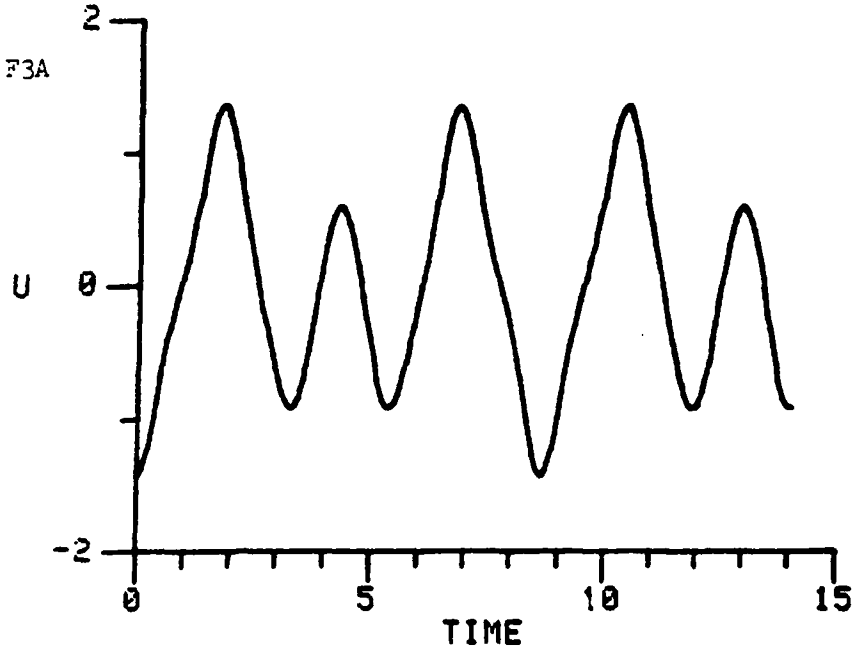



Figure 6.15 CONTROL HISTORY FOR OPTIMAL PERIOD EXTREMA

c. Family F3A




d. Family F5/2A

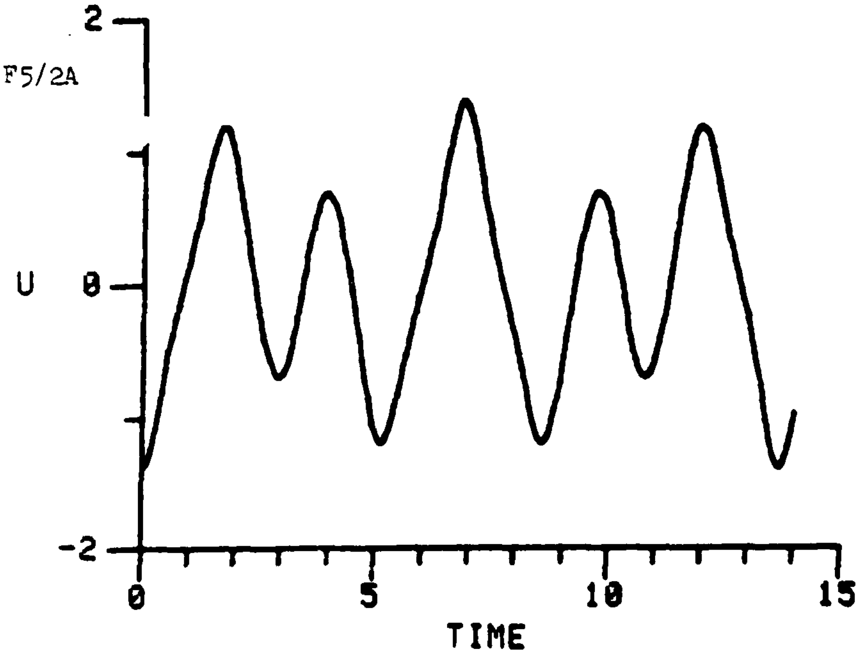



Figure 6.15 CONTROL HISTORY FOR OPTIMAL PERIOD EXTREMA (Continued)

| FAMILY | PERIOD | $x_1(0)$ | $\lambda_2(0)$ | PERFORMANCE |
|---|---|---|---|---|
| Static Equilibrium | 2.1093<br>5.9185 | 0 | 0 | 0 |
| FA | 3.5523 | 0.41173 | 0.14699 | -0.023271 |
| F2A | 4.9815 | 0.37210<br>-0.12028 | 0.12188<br>0.04957 | -0.005052 |
| F3A | 8.6218 | 0.42145<br>0.10215 | 0.14238<br>-0.05961 | -0.012592 |
| F5/2A | 13.6775 | 0.41433 | 0.13837 | -0.009908 |

Table 6.1 Solutions Satisfying Condition for Optimal Period

## 6.4 Sufficiency Condition

The solutions of the previous section must complete one final check for optimality. The Riccati variable associated with each solution must exist over a complete period. The existence of the Riccati variable is examined by two equivalent methods which completes the test of the sufficiency conditions for optimality of the periodic control problem expressed by Theorem 4.2.

The initial conditions for each periodic solution of the Riccati differential equation (2.15) are determined directly from the algebraic expressions (5.38) through (5.41). Two solutions exist for fourth order symmetric systems. They are obtained by integrating the Riccati equation over one period using the two sets of starting conditions just obtained. An equivalent procedure involves evaluating the expression $[\Phi_{11}(t,0) + \Phi_{12}(t,0)P(0)]$ from equation (4.13), and testing for a zero value. The existence of the Riccati variable is implied by no zero values in a one period interval.

The Riccati variable corresponding to the optimum periodic solution of the principal family was found to exist over the one period interval, thus confirming the local optimality of the solution. The time histories of the three elements comprising the Ricatti variable are plotted for both sets of initial conditions in figures 6.16 and 6.17.

Each of the remaining extremal solutions from the previous section evolved from a branch family. At some point in the one period interval of each of these extremal solutions, the corresponding Riccati

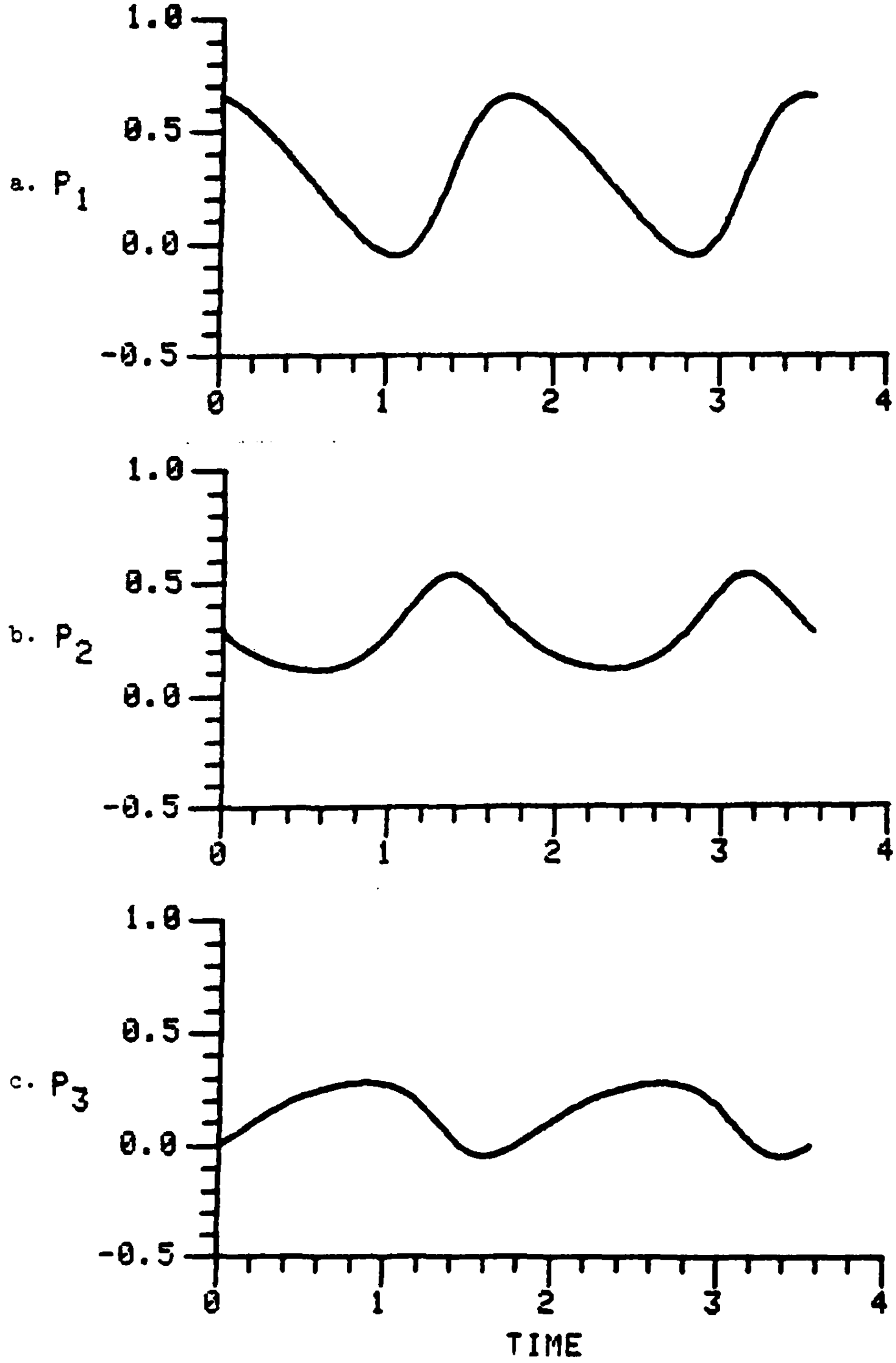



Figure 6.16 SOLUTION TO RICCATI EQUATION (CONVERGENT)

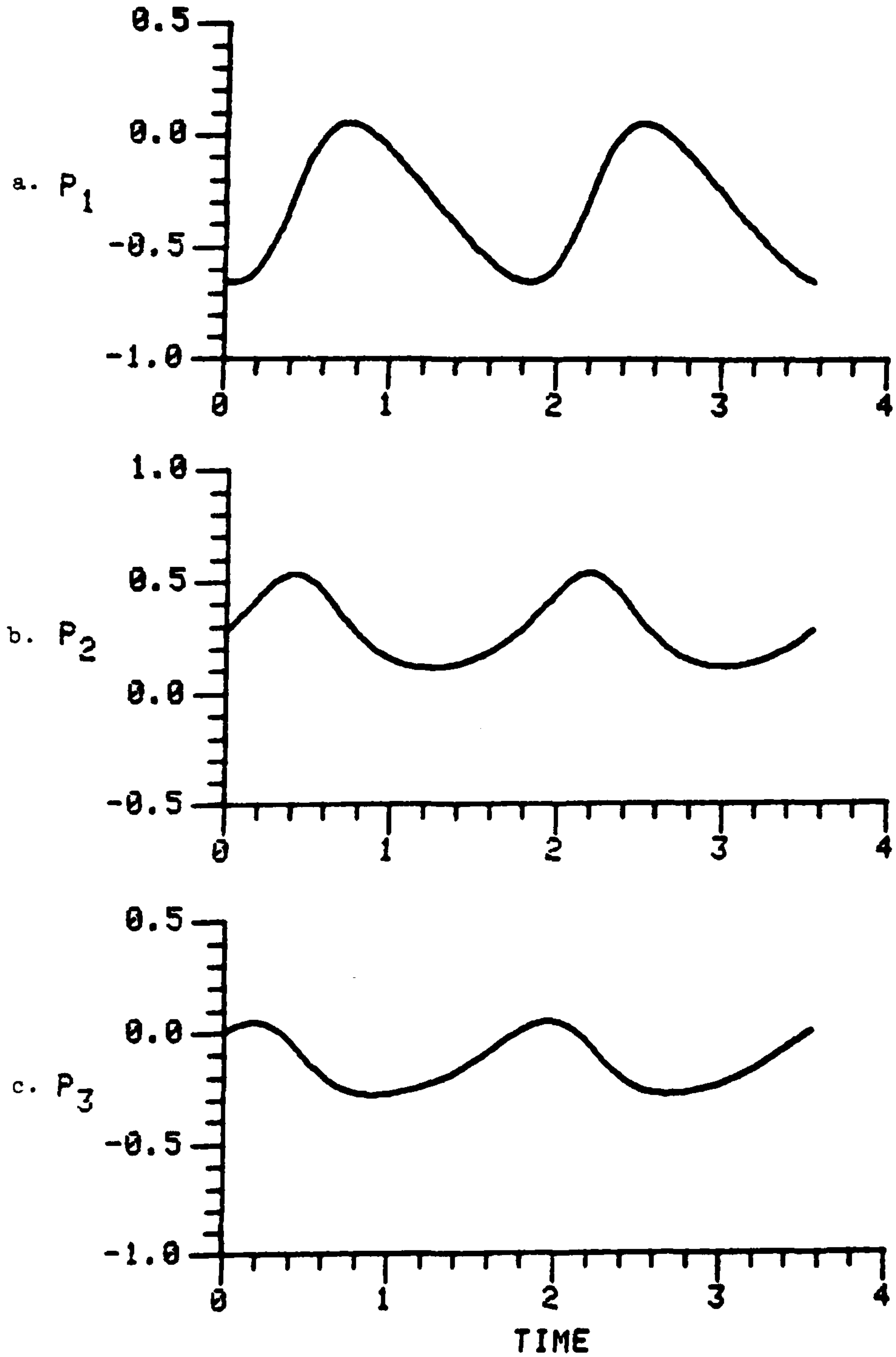



Figure 6.17 SOLUTION TO RICCATI EQUATION (DIVERGENT)

variable escaped (did not exist). Therefore, these solutions are not locally optimizing. As a result, out of the many extremal solutions that have been found for this problem, only the one solution on the principal family satisfies each condition for local optimality of a periodic solution with unspecified period.

The existence of the Riccati variable was also examined for other extrema in the unstable region (stability index, $k > 2$) of the principal family and the branch families. The basic results were the same. Corresponding to extrema of the principal family, the Riccati variable was found to exist during the entire one period interval for the sampling of solutions tested. However, corresponding to extrema of the branch families, the Riccati variable escaped at some time in the interval for each of the solutions tested.

These results indicate that the extremal solutions in the unstable region of the principal family are local optimal solutions with respect to a specified period. The range of periods corresponding to the optimal solutions extend from approximately 2.67 to 4.99. Optimal solutions for specified periods outside this range are as yet unknown. If it could be assumed that n cycles of an optimal periodic solution with respect to a specified period, T, were also optimal for a specified period, nT, there would still be gaps in the range of periods for which optimal solutions exist. It is strongly suspected that nonsymmetric solutions, those whose initial conditions lie outside the symmetric surface investigated in this work, will provide additional local optimal solutions for the problem.

## 6.5 Periodic Regulator

A periodic regulator was developed in chapter four using the neighboring optimal feedback control law, (4.57). In this section, the regulator is implemented using the optimal periodic solution of the previous section as the nominal solution. The regulator is tested by perturbing an initial state and observing the results.

The control diagram for the periodic regulator is shown in figure 6.18a. The plant (or system) for this problem is defined by the dynamic equations (5.2) and (5.3) rewritten below,

$$\dot{x}_1 = x_2, \tag{5.2}$$

$$\dot{x}_2 = u. \tag{5.3}$$

The gains for the regulator are determined from the optimal neighboring feedback control law (4.57) rewritten below,

$$\delta u^o = (H^o_{uu})^{-1}(f^o_u P + H^o_{ux})\delta x. \tag{4.57}$$

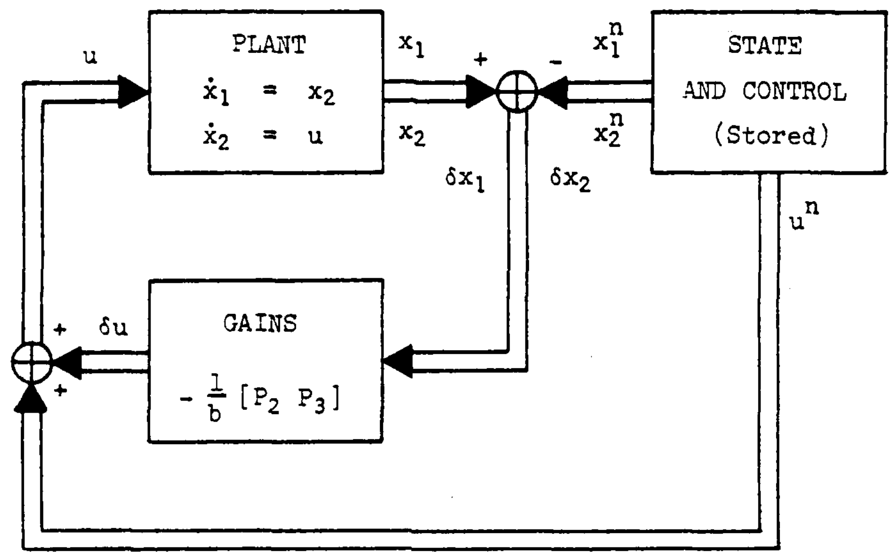
For this problem the gains reduce to

$$G = -\frac{1}{b}[P_2 \; P_3], \tag{6.4}$$

where b = 0.1 and the elements of the Riccati variable are chosen from one of the two possible sets shown in figures 6.16 and 6.17. One of the sets causes a perturbed system to converge back to the nominal path in a limit cycle; the other causes the system to diverge.

The storage block of the regulator is composed of a look-up

a. Control Diagram




b. Error Measurement

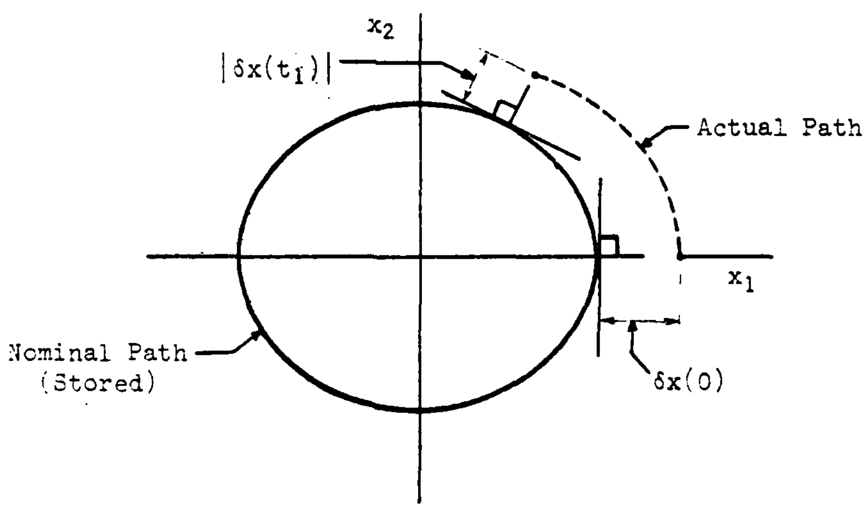



Figure 6.18 PERIODIC REGULATOR

table and a least squares error measurement algorithm. The optimal periodic solution from the previous section is used for determining the nominal values in the table. The terms in the table represent the nominal path plotted in figure 6.14a, the nominal control plotted in figure 6.15a, and the gains determined above. The table spans a full period and each increment represents a time interval. As shown in figure 6.18b, the error measurement algorithm used is time independent. Given the actual state coordinates, the algorithm determines from the table the nominal path coordinates which represent the shortest distance between the actual state and the nominal path. The nominal control and the gains which correspond in the table to the nominal path coordinates just determined are used in this increment of the control cycle.

The regulator is demonstrated by initially perturbing the state and then observing the output of the plant. Two cases are presented for comparative purposes. For both cases the output of the plant (actual path) is represented by a broken line and the nominal path a solid line.

In the first case the periodic regulator is demonstrated by closing the feedback loop and using the set of divergent gains. Cooresponding to these gains are the set of Riccati variable elements shown in figure 6.17. The initial condition, $x_1(0)$, is perturbed by 0.01 per cent. The result, shown in figure 6.19a, illustrates the divergent character of this set of gains.

The final case implements the periodic regulator with its set of convergent gains. The Riccati variables, shown in figure 6.16,

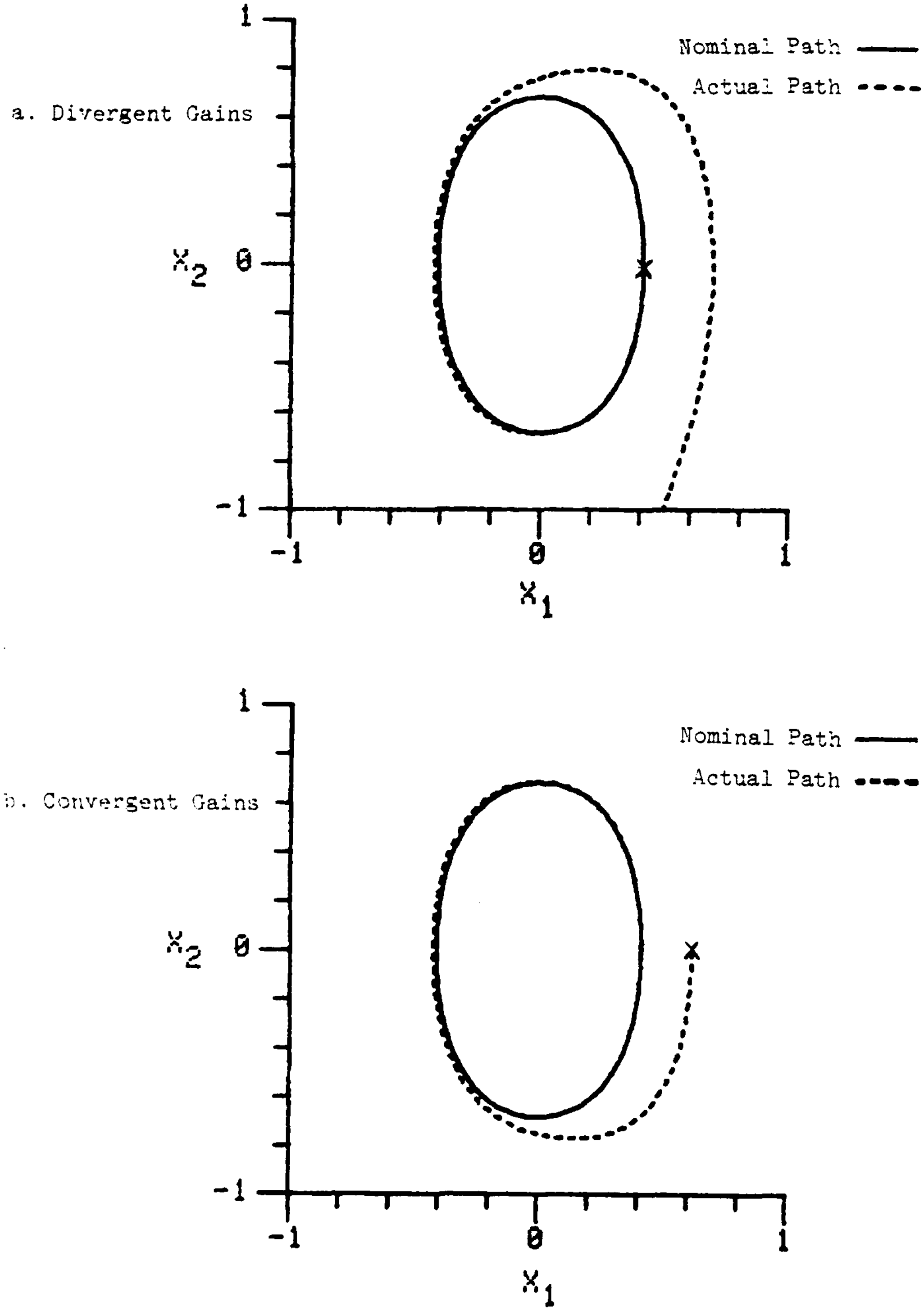



Figure 6.19 PERTURBATION OF OPTIMAL TRAJECTORY WITH PERIODIC REGULATOR

are used to determine these gains. For this case the initial condition is perturbed from the nominal by 50 percent. The strongly convergent behavior of the periodic regulator is shown in figure 6.19b.

# CHAPTER 7

# AN ASYMPTOTIC SOLUTION

An approximate analytical solution to the example problem, defined in chapter five, is determined in this chapter. Using a perturbation technique, the solution is generated in the form of an asymptotic series expansion. Both the dependent and independent variables are expanded about a small perturbation parameter. The success of this technique is very much dependent on the selection of the functional form of the parameter. In the first section, the example problem is reformulated in terms of expanded variables. An expression in the form of a Fourier series expansion for the solution is determined in the second section. Then in the following section, the conditions for an optimal period are applied. This results in asymptotic series expressions for the optimal path, its period, control, and the associated performance index. In the last section of this chapter, the analytical results are compared with linear extrapolations near the static equilibrium solution and with numerical results of the previous chapter in the region near the local minimum operating point.

## 7.1 Formulation of the Problem

The perturbation technique which is used in this chapter to determine an approximate analytical solution for the optimal periodic control problem, equations (5.1) through (5.5), is most

frequently credited to Lindstedt and Poincaré [40,41]. Nineteenth century astronomers, such as Lindstedt (1882), Bohlin (1889), and Gylden (1893), developed techniques to avoid the appearance of secular terms in perturbation solutions of a class of differential equations. Poincaré (1892) proved that the expansions obtained by Lindstedt's technique are asymptotic. In this section, the example problem is reformulated in a similar manner by expanding the independent variable, determining a functional relationship for the expansion parameter, and assuming an asymptotic solution.

To simplify the perturbation analysis, the two point boundary value problem, equations (5.7) through (5.10) is rewritten in the equivalent form of a single fourth order differential equation,

$$b\frac{d^4y}{dt^4} = -\frac{d^2y}{dt^2} + 3\left(\frac{dy}{dt}\right)^2\frac{d^2y}{dt^2} - y, \tag{7.1}$$

where the new variable is defined by $y \equiv x_1$. The periodicity condition is applied to y and its first three derivatives at the boundary points corresponding to an unrestricted period, T. The remaining variables in the original problem may be obtained from the solution to equation (7.1), its derivatives, and the following relationships derived from the Euler-Lagrange equations,

$$x_2 = \frac{dy}{dt}, \tag{7.2}$$

$$\lambda_2 = -b\frac{dy}{dt}, \tag{7.3}$$

$$\lambda_1 = \frac{dy}{dt} - \left(\frac{dy}{dt}\right)^3 + b\frac{d^2y}{dt^2} \,. \tag{7.4}$$

The general, analytical solution to equation (7.1) developed by the perturbation technique is assumed to exist in the form,

$$y(\varepsilon,\tau,\omega) = y_0(\tau,\omega) + \varepsilon y_1(\tau,\omega) + \varepsilon^2 y_2(\tau,\omega) + \ldots, \tag{7.5}$$

where $\tau$ is the expanded independent variable typically defined by the expression

$$t \equiv \tau(1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots), \tag{7.6}$$

$\omega$ is the infinite set, $\{\omega_1, \omega_2, \ldots\}$, of undetermined constant parameters which further scale the independent variable in (7.6), and $\varepsilon$ is the small, perturbation (expansion) parameter. The general solution, (7.5), is obtained by sequentially determining the functional representation of its components related to successive orders of the expansion parameter. In this manner, the approximation may be developed to whatever order is desired.

Before proceeding with the solution development, a particular solution to equation (7.1) about which the general solution is to be asymptotically expanded must be selected, and the functional relationship of the expansion parameter to the system parameters must also be determined. The steady-state solution of the example problem, from the results in section 5.2 of the frequency test, was shown to be optimal for values of the control weighting parameter, $b > \frac{1}{4}$. For values less than this critical value, periodic solutions can be found

which provide better performance. With the objective of developing a general solution for the example problem that represents the class of optimal periodic solutions, an appropriate selection for the particular solution is the steady-state solution coupled with a functional relationship for the expansion parameter that expresses the range for $b < \frac{1}{4}$. The following expression for $\varepsilon$ provides that relationship,

$$b \equiv \frac{1 - \varepsilon^2}{4} , \tag{7.7}$$

where $\varepsilon \leq 1$. The quadratic form of $\varepsilon$ is required here, as will become obvious in the next section, to permit forming expressions that annihilate secular producing terms in the forcing functions of the expansion.

Now an expression for the differential equation (7.1) may be established in terms of the expansion parameter $\varepsilon$, the expanded time variable $\tau$, and the infinite set of constant, undetermined parameters $\omega$. Since $\tau$ is linearly related to t, and the parameters $\varepsilon$ and $\omega$ are constants, the derivatives of $y(\varepsilon,\tau,\omega)$ with respect to t in equation (7.1) are

$$\frac{dy}{dt} = \frac{dy}{d\tau}\frac{d\tau}{dt} , \tag{7.8}$$

$$\frac{d^2y}{dt^2} = \frac{d}{d\tau}\left(\frac{dy}{d\tau}\frac{d\tau}{dt}\right) = \frac{d^2y}{d\tau^2}\left(\frac{d\tau}{dt}\right)^2 , \tag{7.9}$$

$$\frac{d^4y}{dt^4} = \frac{d^4y}{d\tau^4}\left(\frac{d\tau}{dt}\right)^4 . \tag{7.10}$$

The expressions in the above equations associated with the derivative of $\tau$ may be represented by the following infinite series expansion,

$$\frac{d\tau}{dt} = \frac{1}{1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots},$$

$$= 1 - \varepsilon\omega_1 + \varepsilon^2(\omega_1^2 - \omega_2) - \varepsilon^3(\omega_1^3 - 2\omega_1\omega_2 + \omega_3) + \ldots, \quad (7.11)$$

$$\left(\frac{d\tau}{dt}\right)^2 = 1 - \varepsilon 2\omega_1 + \varepsilon^2(3\omega_1^2 - 2\omega_2)$$

$$- \varepsilon^3(4\omega_1^3 - 6\omega_1\omega_2 + 2\omega_3) + \ldots, \quad (7.12)$$

$$\left(\frac{d\tau}{dt}\right)^4 = 1 - \varepsilon 4\omega_1 + \varepsilon^2(10\omega_1^2 - 4\omega_2)$$

$$- \varepsilon^3(20\omega_1^3 - 20\omega_1\omega_2 + 4\omega_3) + \ldots, \quad (7.13)$$

Using the above expressions and equation (7.7), which relates the expansion parameter to the system, permits writing the differential equation (7.1) in the desired form,

$$(1-\varepsilon^2)\frac{d^4y}{d\tau^4}\left(\frac{d\tau}{dt}\right)^4 = -4\frac{d^2y}{d\tau^2}\left(\frac{d\tau}{dt}\right)^2 + 12\left(\frac{dy}{d\tau}\right)^2\frac{d^2y}{d\tau^2}\left(\frac{d\tau}{dt}\right)^4 - 4y, \quad (7.14)$$

where, for brevity, the infinite series expansions have not been explicity included. Analytical expressions for the components of the approximate solution (7.5) are developed in the next section.

## 7.2 Development of Extremal Solutions

An approximate, analytical solution to the differential equation (7.14), which is equivalent to the Euler-Lagrange equations

of the example problem, is developed in this section. This provides a general expression for periodic extrema of the optimal periodic control problem. Then the components of the solution are determined sequentially by substituting the assumed solution, (7.5), into the above differential equation, (7.14), and equating like powers of the expansion parameter, $\varepsilon$, in the resulting infinite series expansion. Operator notation for the derivative with respect to $\tau$, i.e., $D = d/d\tau$, is used in the remainder of this chapter as a convenience in representing lengthy, complex differential equations.

By equating the coefficients of $\varepsilon^0$ in the expansion of equation (7.14), the following differential equation for the $y_0$ component of the general solution is obtained,

$$D^4 y_0 = -4D^2 y_0 + 12(Dy_0)^2 D^2 y_0 - 4y_0. \tag{7.15}$$

As was identified in the previous section, a solution to (7.15) is the steady-state or static solution,

$$y_0 = Dy_0 = D^2 y_0 = D^3 y_0 = 0. \tag{7.16}$$

This also represents the particular solution about which the general solution is expanded.

Differential equations for the remaining components of the general solution are determined in the same manner as those for $y_0$. Each of these differential equations has the following form,

$$(D^4 + 4D^2 + 4)y_i = F_i(y_{i-1}, \ldots, y_0), \tag{7.17}$$

where $F_i$ represents the forcing function of the $i^{th}$ equation in terms of previously determined components. The left hand side of the equation, identical for each unknown component $y_i$, identifies the characteristic equation. The subscript, i, associated with the unknown component, $y_i$, and with the forcing function, $F_i$, also indicates the power of the expansion parameter whose coefficients in the infinite series expansion form the corresponding differential equation (7.17). Forcing functions associated with the unknown components, $y_1$, through $y_4$, are listed below after simplifying the expressions by incorporating the steady-state solution (7.16),

$$F_1 = 0, \tag{7.18}$$

$$F_2 = 4\omega_1 D^2(D^2+2)y_1, \tag{7.19}$$

$$\begin{aligned} F_3 = {} & 4\omega_1 D^2(D^2+2)y_2 + (1-10\omega_1^2+4\omega_2)D^4 y_1 \\ & - (12\omega_1^2-8\omega_2)D^2 y_1 + 12(Dy_1)^2 D^2 y_1, \end{aligned} \tag{7.20}$$

$$\begin{aligned} F_4 = {} & 4\omega_1 D^2(D^2+2)y_3 + (1-10\omega_1^2+4\omega_2)D^4 y_2 \\ & - (12\omega_1^2-8\omega_2)D^2 y_2 + 12(Dy_2)^2 D^2 y_2 \\ & + 24Dy_2 Dy_1 D^2 y_1 - 48\omega_1(Dy_1)^2 D^2 y_1 \\ & + (20\omega_1^3-20\omega_1\omega_2+4\omega_3-4\omega_1)D^4 y_1 \\ & + (16\omega_1^3-24\omega_1\omega_2+8\omega_2)D^2 y_1. \end{aligned} \tag{7.21}$$

Expressions corresponding to any desired number of terms in the general solution (7.5) may be obtained in the same manner. However, the forcing function becomes increasingly more complex as additional terms are added.

To obtain expressions for the components of the general solution, consider first the solution to the characteristic equation,

$$y_i^* = (A_i + C_i\tau)\sin\sqrt{2}\tau + (B_i + D_i\tau)\cos\sqrt{2}\tau \tag{7.22}$$

where * distinguishes this homogeneous solution from the complete solution. In order to satisfy the periodicity conditions, the secular terms in (7.22) must vanish. This requires that the arbitrary constants $C_i$ and $D_i$ must be chosen identically zero.

Now consider the contribution to the component solutions associated with the forcing terms. Combining this particular solution with the previous homogeneous result (7.22) determines the functional expression for respective components of the general solution. Since there is no forcing term associated with the equation for $y_1$, the expression for this component is

$$y_1 = A_1\sin\sqrt{2}\tau + B_1\cos\sqrt{2}\tau. \tag{7.23}$$

The forcing function (7.19) is also zero since $(D^2+2)y_1 = 0$. This yields a similar result for the $y_2$ component

$$y_2 = A_2\sin\sqrt{2}\tau + B_2\cos\sqrt{2}\tau . \tag{7.24}$$

Using the results (7.23) and (7.24) in the expression (7.20) for $F_3$ provides the following relationship,

$$F_3 = -4[4\omega_1^2-1+3(A_1^2+B_1^2)](A_1\sin\sqrt{2}\tau + B_1\cos\sqrt{2}\tau)$$

$$-12(A_1^2-3B_1^2)A_1\sin 3\sqrt{2}\tau - 12(3A_1^2-B_1^2)B_1\cos 3\sqrt{2}\tau. \quad (7.25)$$

Observe that the first term in the forcing function is a solution to the characteristic equation. This will produce a secular term in the solution $y_3$ of equation (7.17). Similar secular producing terms occur in each equation for successive components of y. To satisfy the periodicity condition, the arbitrary constants and undetermined parameters are restricted in a manner that resultant coefficients of $\sin\sqrt{2}\tau$ and $\cos\sqrt{2}\tau$ in each of the corresponding forcing functions are reduced to zero. The restricting equation generated by the forcing function (7.25) is

$$4\omega_1^2 - 1 + 3(A_1^2 + B_1^2) = 0. \quad (7.26)$$

The solution for $y_3$ then becomes

$$y_3 = A_3\sin\sqrt{2}\tau + B_3\cos\sqrt{2}\tau - \frac{3}{64}(A_1^2-3B_1^2)A_1\sin 3\sqrt{2}\tau$$

$$+\frac{3}{64}(3A_1^2-B_1^2)B_1\cos 3\sqrt{2}\tau. \quad (7.27)$$

Using the preceding results for the components $y_1$, $y_2$, and $y_3$ in the expression (7.21) for $F_4$ determines a second restricting equation,

$$3(A_1A_2 + B_1B_2) + 2\omega_1(\omega_1^2 + 2\omega_2) = 0, \quad (7.28)$$

and a solution $y_4$ to equation (7.17),

$$y_4 = A_4\sin\sqrt{2}\tau + B_4\cos\sqrt{2}\tau$$
$$- \frac{3}{128}\left[(\omega_1A_1+6A_2)(A_1^2-B_1^2)-2A_1B_1(\omega_1B_1+6B_2)\right]\sin3\sqrt{2}\tau$$
$$- \frac{3}{128}\left[(\omega_1B_1+6B_2)(A_1^2-B_1^2)+2A_1B_1(\omega_1A_1+6A_2)\right]\cos3\sqrt{2}\tau. \quad (7.29)$$

As this process is continued, one new restricting equation is determined for each additional term evaluated in the expansion. It should be noted that the restricting equation which relates parameters through the $i^{th}$ term of the expansion is determined from the forcing function of the differential equation associated with the coefficients in the expansion of the $i+2^{nd}$ term.

An approximate solution through order $\varepsilon^4$ may now be written in the form of equation (7.5) using the component expressions (7.23), (7.24), (7.27), and (7.29). The resulting expression may be simplified with essentially no loss in generality by imposing the arbitrary boundary condition $\dot{y}(0) = 0$. This fixes the time and phase relationship of the solution, but it does not effect its amplitude or period. As a result the approximate solution becomes

$$y(\varepsilon,\tau,\omega) = \varepsilon B_1\cos\sqrt{2}\tau + \varepsilon^2B_2\cos\sqrt{2}\tau$$
$$+ \varepsilon^3[B_3\cos\sqrt{2}\tau + \frac{3}{64}B_1^3\cos3\sqrt{2}\tau]$$
$$+ \varepsilon^4[B_4\cos\sqrt{2}\tau + \frac{3}{128}(\omega_1B_1^3+6B_1^2B_2)\cos3\sqrt{2}\tau, \quad (7.30)$$

where the expanded time is given through order $\varepsilon^4$ by

$$\tau = t[1 - \varepsilon\omega_1 + \varepsilon^2(\omega_1^2-\omega_2) - \varepsilon^3(\omega_1^3-2\omega_1\omega_2+\omega_3)$$
$$+ \varepsilon^4(\omega_1^4-3\omega_1^2\omega_2+2\omega_1\omega_3-\omega_4)]. \quad (7.31)$$

By rearranging terms in equations (7.30) an expression in the form of a Fourier series is obtained,

$$y(\varepsilon,\tau,\omega) = [\varepsilon B_1 + \varepsilon^2 B_2 + \varepsilon^3 B_3 + \varepsilon^4 B_4]\cos\sqrt{2}\tau$$
$$+ [\varepsilon^3 B_1 + \varepsilon^4 \tfrac{1}{2}(\omega_1 B_1+6B_2)]\frac{1}{64}B_1^2\cos 3\sqrt{2}\tau. \quad (7.32)$$

Continuing the expansion further produces additional cosine terms with arguments that are odd integer multiples of $\sqrt{2}\tau$. An interesting characteristic of the expansion is that the smallest power of $\varepsilon$ in the coefficient of $\cos n\sqrt{2}\tau$ is the odd integer n. This property is commonly referred to as the D'Alembert characteristic [42], and it assures the expansion satisfies the necessary condition for convergence of the series, i.e., the $n^{th}$ term approaches zero as n becomes increasingly large.

It should be noted at this point that the approximate solution to order $\varepsilon^n$ has 2n unknown parameters, $B_k$ and $\omega_k$, where $k = 1, 2, \ldots, n$. However, only n restricting equations relating these parameters can be determined leaving n of them arbitrary. In the next section, n additional relationships are determined by applying the conditions for an optimal period, allowing the parameters to be evaluated.

### 7.3 Optimal Solution with Respect to Period

A general expression was derived in the previous section for extrema of the example optimal periodic control problem. The special transversality condition (5.16) associated with an unrestricted period is applied in this section. First an expansion in powers of $\varepsilon$ is derived for both the variational Hamiltonian and the performance index. Then the expressions are set equal by equating coefficients of like powers of $\varepsilon$. This results in n new relationships for the parameters of the system. Combined with the n restricting equations previously obtained, all 2n parameters in the expression for extrema may be explicitly determined. As a result, the solution (7.32) represents the first n terms of an asymptotic expansion that approximates the optimal solution with respect to an unrestricted period.

The variational Hamiltonaian and the performance index must be evaluated along the extremal path to properly apply the condition for optimal period. The extremal path expressed in terms of the original problem may be determined from the definition $y \equiv x_1$, the extremal solution (7.33), and repeated use of the Euler-Lagrange equations, (5.7) through (5.10). This results in the following series expansions which approximate the extremal path,

$$x_1 = [\varepsilon B_1+\varepsilon^2 B_2+\varepsilon^3 B_3]\cos\sqrt{2}\tau + \varepsilon^3\frac{3}{64}B_1^3\cos 3\sqrt{2}\tau, \tag{7.33}$$

$$x_2 = -\sqrt{2}[\varepsilon B_1+\varepsilon^2(B_2-\omega_1 B_1)+\varepsilon^3(B_3-\omega_1 B_2+(\omega_1^2-\omega_2)B_1)]\sin\sqrt{2}\tau$$
$$-\varepsilon^3\frac{9}{64}\sqrt{2}B_1^3\sin 3\sqrt{2}\tau, \tag{7.34}$$

$$\lambda_1 = -\frac{1}{2}\sqrt{2}[\varepsilon B_1+\varepsilon^2(B_2+\omega_1 B_1)+\varepsilon^3(B_3+\omega_1 B_2+\omega_2 B_1)]\sin\sqrt{2}\tau$$

$$-\varepsilon^3\frac{3}{128}\sqrt{2}B_1^3\sin 3\sqrt{2}\tau, \tag{7.35}$$

$$\lambda_2 = \frac{1}{2}[\varepsilon B_1+\varepsilon^2(B_2-2\omega_1 B_1)+\varepsilon^3(B_3-2\omega_1 B_2+(3\omega_1^2-2\omega_2-1)B_1)]\cos\sqrt{2}\tau$$

$$+\varepsilon^3\frac{27}{128}\, B_1^3\cos 3\sqrt{2}\tau, \tag{7.36}$$

where terms of higher order than $\varepsilon^3$ have been truncated. Using the expression for the optimal control (5.15), the functional representation for the expansion parameter (7.7), and the expression (7.36) provides the series representation of the extrema control

$$u = -2[\varepsilon B_1+\varepsilon^2(B_2-2\omega_1 B_1)+\varepsilon^3(B_3-2\omega_1 B_2+(3\omega_1^2-2\omega_2-1)B_1)]\cos\sqrt{2}\tau$$

$$-\varepsilon^3\,\frac{27}{32}\, B_1^3\cos 3\sqrt{2}\tau. \tag{7.37}$$

Substituting these expressions in equation (5.6), the following general expression for the variational Hamiltonian evaluated along the extremal path is obtained,

$$H = \varepsilon^3 2\omega_1 B_1^2 + \varepsilon^4[4\omega_1 B_1 B_2+(2\omega_2-3\omega_1^2)B_1^2+\frac{9}{8}B_1^4]$$

$$+\varepsilon^5[2\omega_1(B_2^2+2B_1 B_3-\frac{3}{4}B_1^2)-(9\omega_1^2-4\omega_2-\frac{3}{2})B_1 B_2$$

$$+(7\omega_1^3-6\omega_1\omega_2+\omega_3)B_1^2]. \tag{7.38}$$

The first three restricting equations were used to obtain this simplified form of the Hamiltonian. Only terms upto $\varepsilon^4$ in the expressions (7.33) through (7.37) were required to obtain the $\varepsilon^5$ term in (7.38). Note, too, that the functional representation of the variational Hamiltonian is a constant expression, as it should be, when evaluated along an extremal path.

In a similar manner, a general expression for the performance index evaluated along the extremal path is determined by substituting (7.33), (7.34), and (7.37) in the expression for the performance index (5.1) and using the orthogonality characteristics [43] of the sine and cosine in performing the indicated integration. This results in the following constant expression,

$$J = -\varepsilon^4 \frac{3}{8}B_1^4 - \varepsilon^5 \frac{3}{2}B_1^3(B_2-\omega_1 B_1). \tag{7.39}$$

It is interesting that the improvement in the performance is zero to the fourth order.

With expressions for the Hamiltonian and the performance index both evaluated along an extremal path, the condition for optimal period may be applied. Equating the coefficients of like powers of $\varepsilon$ in H and J provides the additional equations needed to specify the remaining arbitrary parameters in the solution. The first three of these optimal period relationships are:

$$\omega_1 B_1^2 = 0, \tag{7.40}$$

$$4\omega_1 B_1 B_2 + (2\omega_2 - 3\omega_1^2) B_1^2 + \frac{3}{2} B_1^4 = 0, \tag{7.41}$$

$$2\omega_1 (B_2^2 + 2B_1 B_3 - B_1^2) - (11\omega_1^2 - 4\omega_2 - 2) B_1 B_2$$
$$+ (9\omega_1^3 - 6\omega_1\omega_2 + \omega_3) B_1^2 = 0. \tag{7.42}$$

The first three restricting equations, (7.26), (7.29), and that from the expansion to $\varepsilon^5$ are rewritten below, using the results, $A_i = 0$ for $i = 1, 2, \ldots$, obtained in the previous section by imposing the boundary condition $\dot{y}(0) = 0$. The simplified restricting equations are:

$$4\omega_1^2 - 1 + 3B_1^2 = 0, \tag{7.43}$$

$$3B_1 B_2 + 2\omega_1 (\omega_1^2 + 2\omega_2) = 0, \tag{7.44}$$

$$\frac{27}{8} B_1^5 - 24 B_1^2 B_3 - 12 B_1 B_2^2$$
$$- 4B_1 (\omega_1^2 + 12\omega_1^2 \omega_2 + 8\omega_1\omega_3 + 4\omega_2^2) = 0. \tag{7.45}$$

The six equations, (7.40) through (7.45), may be solved to obtain values for the constant parameters that determine explicitly through $\varepsilon^3$ the approximate analytical solution that satifies all the first order necessary conditions for an optimum. Equation (7.40) allows the choice of either $B_1 = 0$ or $\omega_1 = 0$. The initial choice leads to $B_1 = B_2 = B_3 = 0$, which results in the static equilibrium solution. For $\varepsilon < \frac{1}{4}$, the static solution is a local maximum. For the second choice, $\omega_1 = 0$, the parameters are determined to be

$$\omega_1 = 0, \quad B_1 = \pm\frac{\sqrt{3}}{3}, \quad B_2 = 0, \quad \omega_2 = -\frac{1}{4},$$

$$\omega_3 = 0, \text{ and } B_3 = \mp\frac{5\sqrt{3}}{192}. \tag{7.46}$$

Note that the parameter $B_3$ has two values. As more terms are included in the expansion, the additional restricting equations and the optimal period relationships become increasingly complex. This results in multiple valued parameters which indicate the possibility of obtaining analytical expressions for numerous locally optimal solutions.

Using the values for the parameters listed in (7.46) leads to the following expressions in terms of $\varepsilon$ and $\tau$ for the optimal path and control,

$$x_1 = \pm[(\frac{\sqrt{3}}{3}\varepsilon - \frac{5\sqrt{3}}{192}\varepsilon^3)\cos\sqrt{2}\tau + \frac{\sqrt{3}}{192}\varepsilon^3\cos 3\sqrt{2}\tau], \tag{7.47}$$

$$x_2 = \mp[(\frac{\sqrt{6}}{3}\varepsilon + \frac{11\sqrt{3}}{192}\varepsilon^3)\sin\sqrt{2}\tau + \frac{\sqrt{3}}{32}\varepsilon^3\sin 3\sqrt{2}\tau], \tag{7.48}$$

$$\lambda_1 = \mp[(\frac{\sqrt{6}}{6}\varepsilon - \frac{21\sqrt{6}}{384}\varepsilon^3)\sin\sqrt{2}\tau + \frac{\sqrt{3}}{1152}\varepsilon^3\sin 3\sqrt{2}\tau], \tag{7.49}$$

$$\lambda_2 = \pm[(\frac{\sqrt{3}}{6}\varepsilon - \frac{37\sqrt{3}}{384}\varepsilon^3)\cos\sqrt{2}\tau + \frac{3\sqrt{3}}{256}\varepsilon^3\cos 3\sqrt{2}\tau], \tag{7.50}$$

$$u = \mp[(\frac{2\sqrt{3}}{3}\varepsilon + \frac{73\sqrt{3}}{273}\varepsilon^3)\cos\sqrt{2}\tau + \frac{3\sqrt{3}}{64}\varepsilon^3\cos 3\sqrt{2}\tau], \tag{7.51}$$

where the expanded time may be expressed as $\tau = (1 + \frac{1}{4}\varepsilon^2)t$. The above solutions are an approximate analytical representation of the local minima associated with the principal family of periodic solutions identified in the previous chapter. In each of these expressions terms of $\varepsilon^4$ and higher have been truncated.

In a similar manner, the constant expressions in terms of $\varepsilon$, for the period, variational Hamiltonian, and performance index that correspond to the previous minimum solution are found to be

$$T = \pi\sqrt{2}(1 - \frac{1}{4}\varepsilon^2), \quad \text{and} \tag{7.52}$$

$$H = J = -\frac{1}{24}\varepsilon^4. \tag{7.53}$$

Terms of order $\varepsilon^6$ and higher have been truncated in the derivation of the last expression.

## 7.4 Verification of Results

The anlytical results developed in this chapter are verified in this section. The functional representation of extremal solutions is compared qualitatively with the numerical results obtained in the previous chapter. In the region near the static equilibrium point the extremal solution is compared to the result of a linear analysis about the same point. Finally, the analytical results obtained in the last section for the optimal solution are compared with corresponding numerical results of the previous chapter.

It is interesting that except for the static equilibrium solution and the the optimal solutions with respect to an unrestricted period, the constants of the periodic extremal solutions, given by equations (7.33) through (7.37), are indeterminant. Examining the extremal solution derived from an expansion up to $\varepsilon^n$ shows that there are 2n constants to be determined. However, there are only n restricting equations that provide relationships between these constants. As a result, n of them may be arbitrarily chosen. This provides for large regions of the phase space to be densely packed with periodic extrema as was shown to be the case for the computational results obtained in the previous chapter. Considering additional higher order terms in the asymptotic expansion greatly complicates the associated relationships for determining the parameters. As the restricting equations and those for optimal period become more complex, the new constants, determined by these equations become multi-valued. This in turn leads to the identification of multiple local extrema, each of which satisfies the optimal period condition. Results obtained numerically in the previous chapter are in agreement.

Now consider the linear region near the static equilibrium point. Choosing the parameters $B_2 = B_3 = 0$ and allowing $B_1 \to 0$ in the expression for the solution restricts it to the region near the static path. Using these values in the restricting equations, (7.43) through (7.45), determines the remaining parameters,

$$\omega_1 = \pm \frac{1}{2}, \quad \omega_2 = -\frac{1}{8}, \quad \omega_3 = \pm \frac{1}{16}. \tag{7.54}$$

The fundamental frequencies defined by the solution given by equations, (7.31) and (7.32) can be written through the $\varepsilon^3$ term as

$$\omega = \sqrt{2}(1 \pm \tfrac{1}{2}\varepsilon + \tfrac{3}{8}\varepsilon^2 \pm \tfrac{5}{16}\varepsilon^3). \tag{7.55}$$

Precisely the same result is obtained for the fast and slow frequencies predicted by the eigenvalues of the linearized Euler-Lagrange equations. The coefficient matrix for this set of linear equations is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{b} \\ -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}, \tag{7.56}$$

evaluated at the static equilibrium point. The eigenvalues of the coefficient matrix are given by

$$\mu^2 = \frac{1}{2b}\,[\pm\sqrt{1-4b} - 1]. \tag{7.57}$$

Using the relationship between the control weighting parameter, b, and the expansion coefficient, $\varepsilon$, then simplifying the result gives the following expression for the frequency,

$$\omega = \sqrt{2}(1 \pm \varepsilon)^{-\frac{1}{2}}. \tag{7.58}$$

where $\mu = i\omega$. Expanding this expression using the generalized binomial theorem gives precise agreement with equation (7.55).

Finally, a comparison is made in table 7.1 between the asymptotic approximation of the optimal solution and the corresponding computational result for several values of the expansion parameter. In the table, $x_1$, and $\lambda_2$ identify the initial values of the optimal periodic solution, where both $x_2$ and $\lambda_1$ have initial value zero. The optimal period is designated T and the performance index, J.

The results of the asymptotic expansion after three terms provides excellent agreement with the computational results for the smaller values of the expansion paramter $\varepsilon$. It provides a good first approximation for large values of $\varepsilon$. The agreement obtained is much better than would be predicted by assuming an accuracy equivalent to the order of $\varepsilon$ in the first truncated term.

| Solution | Analytical Results | Numerical Results | $\varepsilon^2$ |
|---|---|---|---|
| $x_1$ | 0.057699 | 0.057685 | 0.01 |
| $\lambda_2$ | 0.028741 | 0.028734 | 0.01 |
| T | 4.43178 | 4.43174 | 0.01 |
| J | -4.167E-6 | -4.189E-6 | 0.01 |
| $x_1$ | 0.1814 | 0.1809 | 0.1 |
| $\lambda_2$ | 0.0867 | 0.0870 | 0.1 |
| T | 4.332 | 4.327 | 0.1 |
| J | -4.167E-4 | -4.404E-4 | 0.1 |
| $x_1$ | 0.4304 | 0.4117 | 0.6 |
| $\lambda_2$ | 0.1649 | 0.1470 | 0.6 |
| T | 3.776 | 3.552 | 0.6 |
| J | -0.015 | -0.023 | 0.6 |

Table 7.1 Comparative Results

## CHAPTER 8

## CONCLUSIONS AND RECOMMENDATIONS

An attempt has been made in this study of optimal periodic control theory to merge pertinent experience and theory from the fields of analytical dynamics and celestial mechanics with that existing in the optimization and control communities. This has resulted in some important theoretical contributions for this special class of optimal control problems. It has also developed a general characterization of optimal periodic control processes evolving from static equilibrium solutions of this problem. A summary of the more significant results and recommendations for future study are presented in this chapter.

### 8.1 Summary of Conclusions

The principal contributions to the general theory of optimal periodic control are summarized, and particular results of both a numerical and analytical investigation of an optimal periodic control problem are reviewed in this section. There are two key relationships developed in this study which are basic to most of the theoretical results. One is an algebraic equation which relates the initial conditions for periodic solutions to the Riccati differential equation with the elements of the monodromy matrix. The other is a particular similarity transformation of the monodromy matrix which results in a partitioning of its eigenvalues into the two submatrices on its principal diagonal.

A relationship between real values of the Riccati variable and the eignevalues of the monodromy matrix is derived by exploiting the symplectic properties of the monodromy matrix, and the special properties of its eigenvalues both preserved through the similarity transformation. The important conclusion derived from this relationship is that locally optimal periodic solutions of the control problem are generally unstable. If an eigenvalue associated with an optimal periodic solution lies on the unit circle at other than 1 or -1, it must occur in sets of double conjugate pairs. At the critical points 1 and -1, the eigenvalues must occur in coupled pairs.

A deficiency is corrected in the previously existing sufficient conditions for local optimality of periodic control problems with fixed periods. The specific correction identifies that two unity eigenvalues are associated with all extrema of the optimal periodic control problem. This is a property of the monodromy matrix of a Hamiltonian system such as the Euler-Lagrange equations in the optimal periodic control problem. The sufficiency conditions are then extended to include the class of problems for which the period is unrestricted. The neighboring optimal control law, derived in the above process, is used to develop a periodic regulator. It optimizes the return path for small perturbations from the nominal path in a manner analagous to the static regulator.

Two very interesting and simple results are derived from the preceding general developments for symmetric, fourth order optimal control problems. An inequality expression involving only the trace of the monodromy matrix provides a necessary condition for optimal

periodic solutions. The second result provides an explicit expression in terms of elements of the monodromy matrix for the initial conditions of periodic solutions to the Riccati differential equation. This is used to show that only two periodic solutions of the Riccati equation can exist for such a system.

In the second half of this study a particular optimal control problem, constructed to characterize optimal periodic solutions, is investigated both numerically and analytically. The remarkable complexity of its solutions is characterized by the numerical results. An infinity of extremal solutions are found forming continuous, one-parameter families that densely pack large regions of the phase space. They intersect at common solutions which may be determined by a measure of their stability. Characteristics, such as the number of arcs in a period, are identified to distinguish solutions of different families. The computer program used in this study exploits these distinguishing characteristics and obtains extremal solutions by tracing families in a systematic and predictable manner. Multiple solutions are found that satisfy the necessary condition for an optimal period. The Riccati variable associated with these solutions is found to exist over the full period for only the solution associated with the principal family. However, it is conjectured that other solutions exist that also satisfy each condition for local optimality. Similar results for local optima with specified period are determined.

The optimal periodic control problem is also analyzed using a perturbation technique which results in reformulating the problem as an asymptotic expansion. General expressions for the solution's state,

period and performance index is derived in terms of the independent variable, time, and an expansion parameter. The expressions are in the form of a Fourier series with its fundamental frequency and coefficients represented by asymptotic expansions. A comparison of the analytical results with relatively few terms to the numerical results is quite good.

## 8.2 Recommendations for Future Study

There are several avenues along which future effort might be directed. The principal problem that motivated this study was the apparent lack of convergence of typical first order numerical optimization techniques for particular optimal periodic control problems. The development of efficient computational algorithms for this class of optimization problem is an important and fruitful path. Higher order, more complex systems must be addressed such as in the recent work on a hypersonic cruise cycling trajectory, by D. Walker [44].

The numerical investigation initiated in this study is virtually limitless. Solutions with initial conditions lying in only one region of the initial condition surface of symmetry was investigated in this study. Nonsymmetric solutions exist and theory is available to predict intersections with symmetric solutions. Also, irregular families that have no interconnection at all with the principal families have been shown to exist for other dynamic problems [25]. Additional optimal solutions with greater improvement in performance may be found. Another possible avenue of research is the investigation of quasi-periodic solutions which may provide better performance for

the infinite time problem (not addressed in this study) which is a generalization of a periodic solution of increasingly large period.

The further development of the asymptotic series expansions should identify additional optimal solutions. It might also lead to the generalization of the frequency test to determine optimality for arbitrary extremal trajectories.

## APPENDIX A

## A SOLUTION TO THE RICCATI DIFFERENTIAL EQUATION

By solving an appropriate set of linear differential equations, a solution to the following Riccati differential equation may be obtained,

$$\dot{P}(t) = -A^T(t)P(t)-P(t)A(t)-P(t)B(t)P(t)+C(t). \qquad (A.1)$$

The following is essentially a restatement of a Lemma and proof presented by Brockett [36], pages 155-157.

Given the set of 2n linear differential equations,

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A(t) & B(t) \\ C(t) & -A^T(t) \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}, \qquad (A.2)$$

let its transition matrix, $\Phi$, be partitioned into four n-square sub-matrices as follows,

$$\Phi(t,t_0) = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}. \qquad (A.3)$$

Then a solution to equation (A.1) is

$$P(t) = [\Phi_{21} + \Phi_{22}P(t_0)][\Phi_{11} + \Phi_{12}P(t_0)]^{-1} \qquad (A.4)$$

when the indicated inverse exists.

In order to show (A.4) is in fact a solution to (A.1), first examine the boundary conditions. Applying $\Phi(t_0,t_0) = I$ to equation (A.4) gives $P(t_0) = P(t_0)$. Clearly the boundary conditions are satisfied. Substituting (A.4) into (A.1) and reducing the result to an identity would complete the verification. To simplify the algebra involved in doing this, first define the n-square matrices X and $\Lambda$ by

$$\begin{bmatrix} x(t) \\ \\ \Lambda(t) \end{bmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} I \\ \\ P(t_0) \end{bmatrix}. \tag{A.5}$$

Since $\Phi$ is a solution to equation (A.2), the linear combination of $\Phi$ in (A.5) is also; hence,

$$\begin{bmatrix} \dot{X}(t) \\ \\ \dot{\Lambda}(t) \end{bmatrix} = \begin{bmatrix} A(t) & B(t) \\ \\ C(t) & -A^T(t) \end{bmatrix} \begin{bmatrix} X(t) \\ \\ \Lambda(t) \end{bmatrix}. \tag{A.6}$$

Using X and $\Lambda$ from (A.5), the expression for the solution P(t) given by (A.4) can be written,

$$P(t) = \Lambda(t)X^{-1}(t). \tag{A.7}$$

Since the time derivative of the inverse of a matrix may be expressed as

$$\dot{X}^{-1}(t) = -X^{-1}(t)\dot{X}(t)X^{-1}(t), \tag{A. 8}$$

the time derivative of equation (A.7) is

$$\dot{P}(t) = \dot{\Lambda}(t)X^{-1}(t) - \Lambda(t)X^{-1}(t)\dot{X}(t)X^{-1}(t) \; . \tag{A.9}$$

Using the expressions for $\dot{X}(t)$ and $\dot{\Lambda}(t)$ from (A.6), and for $P(t)$ from (A.4), the expression in (A.9) is reduced to the expression in (A.1) as follows,

$$\begin{aligned} \dot{P}(t) = C(t)X(t)X^{-1}(t) &- A^T(t)\Lambda(t)X^{-1}(t) \\ &-\Lambda(t)X^{-1}(t)A(t)X(t)X^{-1}(t) \\ &-\Lambda(t)X^{-1}(t)B(t)\Lambda(t)X^{-1}(t), \end{aligned} \tag{A.10}$$

$$\dot{P}(t) = C(t) - A^T(t)P(t) - P(t)A(t) - P(t)BP(t). \tag{A.11}$$

Thus it has been shown that the expression in (A.4) is the *general* solution to the Riccati equation (A.1).

## APPENDIX B

## THE MONODROMY MATRIX OF A SYMMETRIC SYSTEM

The following derivation of the monodromy matrix for a symmetric, fourth order system is an extraction from the notes of Dr. Roger A. Broucke.

Assume a fourth order, dynamical system of the form

$$\ddot{x} = U_x(x,y), \qquad x(t+T) = x(t),$$
$$\ddot{y} = U_y(x,y), \qquad y(t+T) = y(t), \tag{B.1}$$

where x and y are scalars and U(x,y) is a potential function. Assume further that the system is symmetric, satisfying

$$x(-t) = x(t); \qquad \dot{x}(-t) = -\dot{x}(t),$$
$$y(-t) = -y(t); \qquad \dot{y}(-t) = \dot{y}(t), \tag{B.2}$$

such that the system is invariant. This implies the following relationship,

$$\dot{U}(x,y) = U_x(x,y)\dot{x}(t) + U_y(x,y)\dot{y}(t) = 0. \tag{B.3}$$

Using equations (B.1) and (B.2) in equation (B.3) results in the following useful expressions,

$$U_x(x,y) = U_x(x,-y),$$
$$U_y(x,y) = -U_y(x,-y). \tag{B.4}$$

The variational equations of (B.1) can be written as

$$\delta\ddot{x} = U_{xx}\delta x + U_{xy}\delta y,$$

$$\delta\ddot{y} = U_{yx}\delta x + U_{yy}\delta y, \tag{B.5}$$

which applying (B.1) and (B.2) again require that

$$U_{xx}(x,y) = U_{xx}(x,-y),$$

$$U_{xy}(x,y) = -U_{xy}(x,-y),$$

$$U_{yx}(x,y) = -U_{yx}(x,-y),$$

$$U_{yy}(x,y) = U_{yy}(x,-y). \tag{B.6}$$

Rewriting equations (B.5) in the form of four first order equations, where

$$v_1 = \delta x,$$

$$v_2 = \delta y,$$

$$v_3 = \delta\dot{x},$$

$$v_4 = \delta\dot{y}. \tag{B.7}$$

This results in the following set of equations

$$\dot{v}_1 = v_3,$$

$$\dot{v}_2 = v_4,$$

$$\dot{v}_3 = U_{xx}v_1 + U_{xy}v_2,$$

$$\dot{v}_4 = U_{yx}v_1 + U_{yy}v_2, \tag{B.8}$$

which in vector notation is

$$\dot{V} = A\,V, \tag{B.9}$$

where V is the vector composed of $v_1$, $v_2$, $v_3$, and $v_4$; and A is defined as

$$A \equiv \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ U_{xx} & U_{xy} & 0 & 0 \\ U_{yx} & U_{yy} & 0 & 0 \end{bmatrix}. \tag{B.10}$$

Using the symmetry properties (B.6), it is easy to verify that

$$A(-t) = -BA(t)B, \tag{B.11}$$

where

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{B.12}$$

Note that B has the following properties:

$$B^T B = I; \quad B^T = B^{-1} = B. \tag{B.13}$$

Consider the transition matrix $\Phi(t,0)$ for this system. It satisfies the variational equations (B.8)

$$\dot{\Phi}(t,0) = A(t)\Phi(t,0). \tag{B.14}$$

Replacing t by -t and, using the previous symmetry relations, equation (B.14) becomes

$$\dot{\Phi}(-t,0) = -A(-t)\Phi(-t,0), \tag{B.15}$$

which using equation (B.11) can be written

$$\dot{\Phi}(-t,0) = BA(t)B\Phi(-t,0) \quad \text{or}$$

$$B\dot{\Phi}(-t,0) = A(t)B\Phi(-t,0). \tag{B.16}$$

This shows $B\Phi(-t,0)$ is a fundamental solution matrix of equation (B.8) and must be related to the transition matrix (or to any other fundamental solution matrix) by post multiplication of a constant matrix,

$$B\Phi(-t,0) = \Phi(t,0)C. \tag{B.17}$$

From the initial conditions the constant matrix C must be B, therefore the following relationship may be written,

$$\Phi(-t,0) = B\Phi(t,0)B. \tag{B.18}$$

Finally, these symmetry properties applied at a particular value of t, equal to one full period T, are considered. From properties of the mondromy matrix, the following relation may be written,

$$\Phi(t+T,0) = \Phi(t,0)\Phi(T,0). \tag{B.19}$$

When t = -T, equation (B.19) reduces to

$$I = \Phi(-T,0)\Phi(T,0). \tag{B.20}$$

From equations (B.20) and (B.18), the following may be written,

$$\Phi^{-1}(T,0) = \Phi(-T,0) = B\Phi(T,0)B. \tag{B.21}$$

Since the monodromy matrix is symplectic,

$$\Phi^{-1}(T,0) = -K\Phi^{T}(T,0)K, \tag{B.22}$$

where K is the fundamental symplectic matrix. Combining equations (B.21) and (B.22) gives

$$\begin{aligned} -K\Phi^{T}(T,0)K &= B\Phi(T,0)B, \\ \Phi^{T}(T,0) &= -KB\Phi(T,0)BK, \\ \Phi^{T}(T,0) &= (BK)\Phi(T,0)(BK), \end{aligned} \tag{B.23}$$

where, from the definitions of B and K,

$$BK = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}. \tag{B.24}$$

Forming the identities which result from equating the sixteen elements of the resulting two matrices from equation (B.23) establishes six nontrivial symmetry relationships for the monodromy matrix, $\Phi(T,0)$,

$$\begin{aligned} \phi_{33} &= \phi_{11}, & -\phi_{34} &= \phi_{21}, \\ -\phi_{43} &= \phi_{12}, & \phi_{44} &= \phi_{22}, \\ -\phi_{32} &= \phi_{41}, & \phi_{14} &= \phi_{23}. \end{aligned} \tag{B.25}$$

From these relationships, the monodromy matrix for a symmetric, fourth order system may be written in the following form,

$$\Phi(T,0) = \begin{bmatrix} a & b & e & f \\ c & d & -f & g \\ h & i & a & -c \\ -i & j & -b & d \end{bmatrix}. \tag{B.26}$$

This result also shows that only the first two columns of the monodromy matrix need be computed to obtain the trace of the matrix.

## APPENDIX C

## CLASSIFICATION OF CRITICAL SOLUTIONS

Critical points (solutions) of the first and second kinds are identified and classified in this section for the principal family of solutions determined in the numerical investigation of chapter six. This is directly attributable to the work of Hènon [27] and Contopoulos [26], studying the stability characteristics of fourth order periodic systems. No attempt is made to develop or reproduce the associated theory; only the classification by type of the critical solutions of the sample problem and the significance are expressed.

On the principal family, there are four critical points of the first kind ($k_c = +2$); 1A, 1B1, 1B2, and 1B3. Three critical points of the second kind ($k_c = -2$); 2A, 2B1, and 2B2 also exist. The location of each of these points is shown in figure 6.3 on the initial condition plot of the principal family. In table C.1 the initial conditions associated with the critical solutions are identified. The Hènon matrix and resulting classification type are specified as well. The Hènon matrix is essentially the result of reducing the order of the system by two. Corresponding to the off-diagonal elements of this matrix, different types of exchange of stability are identifiable. For the critical points of the first kind, each is associated with a particular Hènon type. The critical

| SOLUTION IDENTIFIER | STABILITY INDEX | INITIAL CONDITIONS $x_1$ | $\lambda_2$ | HÉNON MATRIX | TYPE |
|---|---|---|---|---|---|
| 1A | 2 | .25755543 | .15021383 | $\begin{bmatrix} 1 & -6.32 \\ 0 & 1 \end{bmatrix}$ | 1 |
| 1B1 | 2 | .37068574 | .06118671 | $\begin{bmatrix} 1 & -0.85 \\ 0 & 1 \end{bmatrix}$ | 3 |
| 1B2 | 2 | .38910618 | .06611312 | $\begin{bmatrix} 1 & 0 \\ -0.56 & 1 \end{bmatrix}$ | 4 |
| 1B3 | 2 | .44846505 | .08778399 | $\begin{bmatrix} 1 & -7.58 \\ 0 & 1 \end{bmatrix}$ | 1 |
| 2A | -2 | .16468522 | .12070592 | $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ | D' R1 |
| 2B1 | -2 | .25454014 | .03621871 | $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ | D' |
| 2B2 | -2 | .43382764 | .08113393 | $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ | D' |

Table C.1 Classification of Critical Solutions

points of the second kind are all of the type not classified by Hènon (both off-diagonal elements are zero). Contopoulos' classification type is identified by these cases.

The significance of the classification types is illustrated in figure C.1 for critical solutions of both first and second kinds. The solid line identifies unstable solutions; the dashed line indicates the stable solutions; and the dotted line represents nonsymmetric solutions that do not lie in the symmetric initial condition surface. Each critical solution is found at the intersection of families of solutions or at the stability transition point.

Solutions 1A and 1B3 are extremum in energy (Hamilitonian) and represent only the transition between stable and unstable solutions on the principal family. No bifurcation occurs at this point as depicted in figure C.1a. Solution 1B1 is a bifurcation point and is represented in figure C.1b. The final critical solution of the first kind, 1B2, is shown in figure C.1c. The intersection of a nonsymmetric family with the principal family is predicted at this point.

The critical solutions of the second kind are all of the type D', as classified by Contopoulos. The upper half solutions (H > 0) were not investigated and, therefore, are not further classified. The solution 2A, shown in figure C.1d, is identified as a trifurcation point. The two symmetric families have been traced. The nonsymmetric family of solutions has not yet been found.

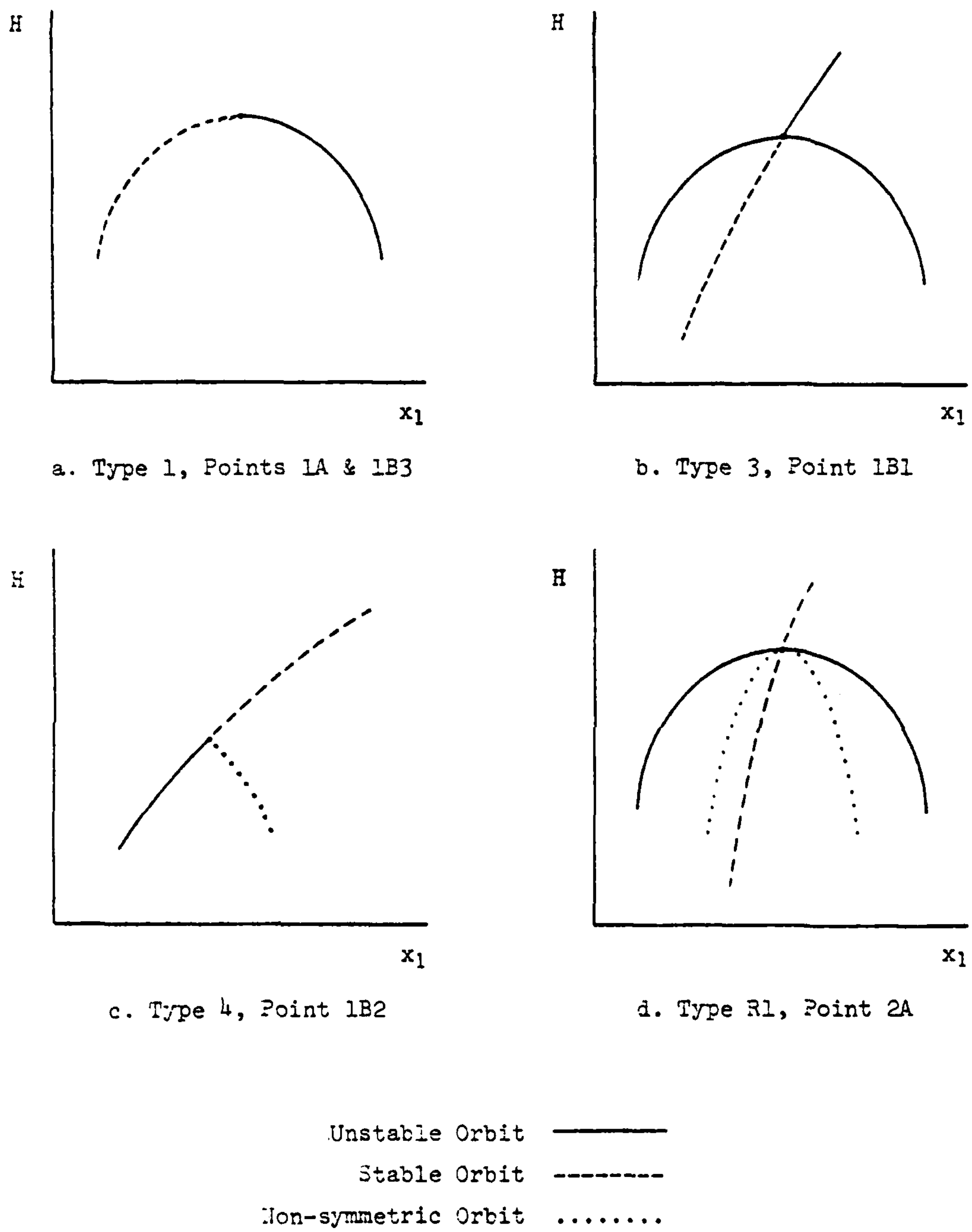



Figure C.1 CLASSIFICATION OF CRITICAL POINTS

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